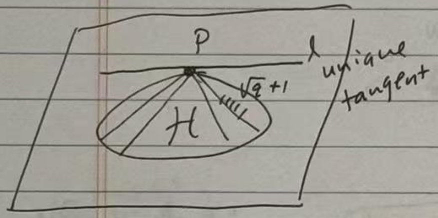
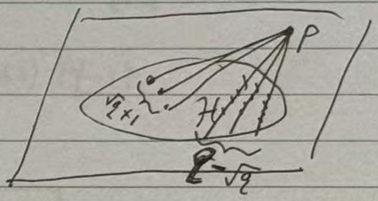
A graph arising from Hermitian Unitals

$PG(2,q)$, q = a square.





We define a graph Γ as follows:

The vertices of Γ are the secant lines to $\mathcal{H}$

$$|V(\Gamma)| = q(q-\sqrt{q}+1)$$

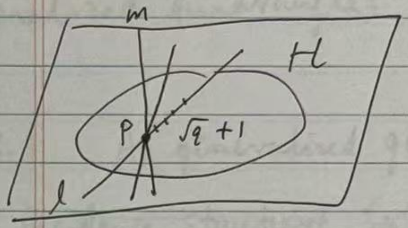
total # of line is q^2+q+1

# of tangents $= q\sqrt{q}+1$

Let l and m be two secant lines to $\mathcal{H}$.

$$l \overset{\Gamma}{\sim} m \iff l \cap m \in \mathcal{H}.$$

Claim: Γ is a $\left(v = q(q-\sqrt{q}+1),\ k = (q-1)(\sqrt{q}+1),\ \lambda = 2q-2,\ \mu = (\sqrt{q}+1)^2\right)$ – SR



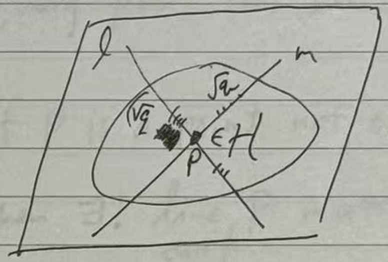
过p有唯一的一条切线

$$q+1-\underbrace{1}_{\text{tang line}}-\underbrace{1}_{l\text{本身}} = q-1$$

valency

$$\therefore\ k=(\sqrt{q}+1)(q-1).$$

Now 计算 λ:



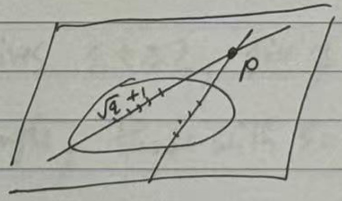
$$\sqrt{q}\cdot\sqrt{q} = q$$

过p有唯一的一条切线

$$q+1-\underbrace{1}_{\text{tang line}}-2 = q-2$$

$$\lambda = q+q-2 = 2q-2$$

Now 计算 μ:



$$(\sqrt{q}+1)^2 = \mu.$$

So we have shown that Γ is a (v,k,λ,μ)-SRG; its eigenvalues are $k=(q-1)(\sqrt{q}+1)$, $r=q-\sqrt{q}-2$, $s=-\sqrt{q}-1$

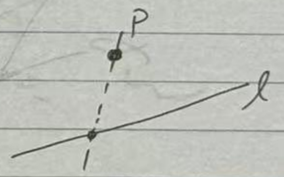
Generalized quadrangles

Def. 3.1 A generalized quadrangle is a point-line incidence structure satisfying the following:

(GQ1): Every line has $s+1$ pt for some $s \geq 1$

(GQ2): Every point lies on $t+1$ lines for some $t \geq 1$

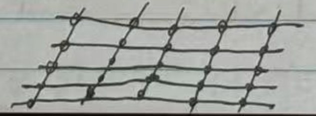
(GQ3): If P is a point not on a line l, then $\exists!$ line through P meeting l



We call (s,t) the order of the GQ, or if $s=t$ we say simply a GQ of order s.

Usually one requires $s,t \geq 2$, giving the so-called thick generalized quadrangle; those with $s=1$ or $t=1$ are thin.

Indeed a generalized quad of order $(s,1)$ is just a 2-net of order $s+1$, i.e., a grid with $(s+1)^2$ points and $2(s+1)$ lines



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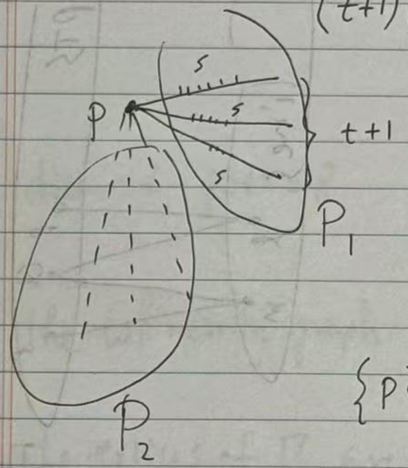
Theorem 3.2. A GQ of order (s,t) has

exactly $(s+1)(st+1)$ points

$(t+1)(st+1)$ lines

proof:



$\{p\} \cup \mathcal{P}_1 \cup \mathcal{P}_2$

$|\mathcal{P}_1| = s(t+1).$

Count (p', Q), $p' \in \mathcal{P}_1$, $Q \in \mathcal{P}_2$ $\quad$ p' & Q are collinear

in two ways.

Start with Q: $\quad |\mathcal{P}_2|(t+1)$

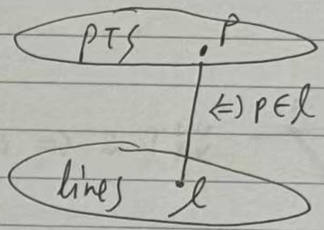
Start with p': $\quad |\mathcal{P}_1| \cdot st$

$$\Rightarrow |\mathcal{P}_2| = \frac{|\mathcal{P}_1| st}{t+1} = s^2 t$$

Hence the # of pts is $1 + s(t+1) + s^2 t = (s+1)(st+1)$

Two Graphs Associated with a $GQ(s,t)$:

Γ_1: the incidence graph



diam $= 4$, girth $= 8$

Γ_2: the Collinearity graph

The vertices of Γ_2 are the points of the GQ.

$$P \sim Q \iff P \text{ and } Q \text{ are Collinear.}$$

Claim: Γ_2 is an SRG

Pf: Let A be the adjacency matrix of Γ_2.

$$V(\Gamma_2) = \{P_1, P_2, \cdots, P_{(st+1)(s+1)}\}$$

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } P_i \sim P_j \\ 0 & \text{if } P_i \not\sim P_j \\ & \text{or } i = j \end{cases}$$

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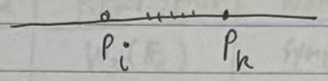
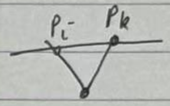
Valency $= k = s(t+1)$

$p_i \sim p_k$

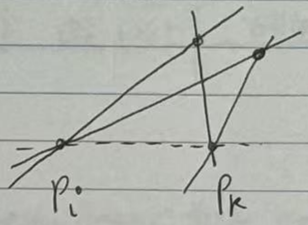


注意 no triangles $\Rightarrow$ 不可能



$\lambda = s-1$

$p_i \nsim p_k$



$\mu = t+1$

$$A^2 - (s-t-2)A - (t+1)(s-1)I = (t+1)J$$

即 Γ_2 is a $\big((st+1)(s+1),\ s(t+1),\ \lambda = s-1,\ \mu = t+1\big)$ — SRG

eigenvalues: $s(t+1)^1$, $(-t-1)^{m_1}$, $(s-1)^{m_2}$,

$$1 + m_1 + m_2 = (st+1)(s+1)$$

$$s(t+1) - m_1(t+1) + m_2(s-1) = 0$$

$$\Rightarrow (s+t) \mid st(s+1)(t+1).$$

Classical Generalized quadrangles

k: v.s. dim	Polar Space	Name	order (s,t)
4	$W_3(\mathbb{F}_q)$	Symplectic	(q,q)
4	$U_3(\mathbb{F}_q)$	unitary	$(q,\sqrt{q})$
5	$U_4(\mathbb{F}_q)$	unitary	$(q,q\sqrt{q})$
4	$Q_3^+(\mathbb{F}_q)$	hyperbolic	$(q,1)$
5	$Q_4(\mathbb{F}_q)$	parabolic	(q,q)
6	$Q_5^-(\mathbb{F}_q)$	Elliptic	(q,q^2)

Def. 4.1. A σ-sesquilinear form on $V(n,q) = V$ is a map

$$\beta : V \times V \to \mathbb{F}_q \quad \text{such that}$$

(i) $\beta(u+w, v) = \beta(u,v) + \beta(w,v)$

(ii) $\beta(u, w+v) = \beta(u,w) + \beta(u,v)$

(iii) $\beta(au, bv) = ab^\sigma \beta(u,v)$, where σ is an automorphism of $\mathbb{F}_q$.

If $\sigma = 1$, then β is called bilinear.

A form is <u>degenerate</u> if $\exists$ a $w \neq 0$ Such that $\beta(u,w) = 0$ for all $u \in V$ or $\beta(v,u) = 0$ for all $u \in V$.

A form β is called reflexive if $\beta(u,v)=0 \Rightarrow \beta(v,u)=0$.

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Thm 4.2. (Birkhoff and von Neumann)

Let β be a nondeg σ-sesquilinear reflexive form on $V=V(n,q)$.

Up to a scalar factor, β is one of the following types:

(i) Alternating : $\beta(u,u)=0 \ \forall$ all $u \in V$

(ii) Symmetric : $\beta(u,v)=\beta(v,u)$ for all $u,v\in V$

(here $\sigma=1$)

(iii) Hermitian : $\beta(u,v)=\beta(v,u)^{\sigma}$, for all $u,v\in V$

where $\sigma^2=1$, $\sigma\neq 1$.

Classical Polar Spaces:

Let β be a nondeg σ-sesquilinear form on $V(n,q)=V$.

A vector $u\in V$ is called isotropic if $\beta(u,u)=0$

A subspace U is called totally isotropic if $\beta(u,v)=0 \ \forall u,v\in U$

The polar space associated to a σ-sesquilinear form on V is the geometry whose points, lines, planes, ... are the totally isotropic subspaces of $V(n,q)$ of rank 1, 2, 3, ••• .

Alternating Forms (Symplectic Polar Spaces)

Let β be an nondeg alternating form on $V(n, \mathbb{F}_q)$ with the property that $\beta(u,u) = 0, \ \forall u \in V$.

(1) $b(u,v) = -b(v,u), \ \forall u, v \in V$ ✓

(2) A maximum totally isotropic subspace with resp to β defined on $V(n, \mathbb{F}_q)$ has dim $\frac{1}{2}n$. $\quad U \subseteq U^{\perp}$

(Hence $n = \text{even} = 2r$)

(3) A non deg alternating form β on $V(n, \mathbb{F}_q)$ is, with resp to a suitable basis,

$$\beta(u,v) = u_1v_2 - u_2v_1 + u_3v_4 - u_4v_3 + \cdots$$

Or the Gram matrix of β (with resp to a suitable basis) is

$$\left[\begin{array}{c|c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ \hline 0 & & \ddots \end{array}\right]$$

$W_3(\mathbb{F}_q)$, or $W(3,q)$

$W_3(\mathbb{F}_q)$ is the point-line incidence structure where

$\mathcal{P}$ = the totally isotropic pts of $V(4,q)$ w.r.t β (nondeg. alternating form)

since $\beta(u,u)=0$ $\forall u$ $\searrow$ = the set of all pts of $PG(3,q)$.

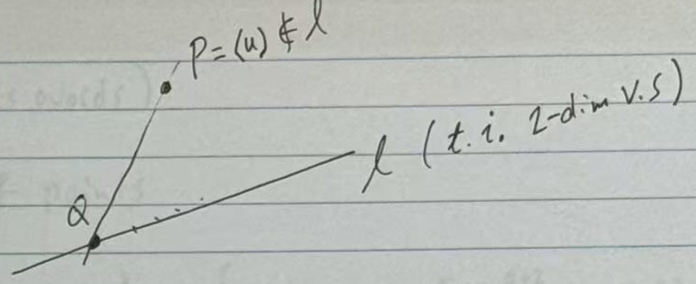
$\mathcal{L}$ = the set of t.i. 2-dim subspaces of $V(4,q)$.

Claim: $\underbrace{(\mathcal{P},\mathcal{L})}_{W_3(\mathbb{F}_q)=}$ is a $GQ(q,q)$.

pf: (GQ1) A line of $W_3(\mathbb{F}_q)$ is a totally isot 2-dim subspace U of $V(4,q)$; this U has $\frac{q^2-1}{q-1}=q+1$ 1-dim subspaces, each of them is totally isotropic (since β is alternating).

(GQ2) Let $P\in\mathcal{P}$ be a point of $W_3(\mathbb{F}_q)$, say $P=\langle u\rangle$.
Since $\langle u\rangle$ is t.i., we have $\langle u\rangle\subseteq u^\perp \Rightarrow \dim u^\perp/\langle u\rangle = 2$
Hence there are $\frac{q^2-1}{q-1}$ lines (t.i.) through P.

(GQ 3)



$\dim u^{\perp} = 3$, $\underset{\text{v.s}}{\underline{\dim}}\, \ell = 2 \Rightarrow \dim(\underline{u^{\perp} \cap \ell}) = 1$

Let $Q = \langle w \rangle = u^{\perp} \cap \ell$

Hence we have shown that $W_3(q)$ is a $GQ(q,q)$.

Substructures in $GQ(s,t)$:

An ovoid of a $GQ(s,t)$ is a set of points $\mathcal{O}$ with the property that each line contains exactly one point of $\mathcal{O}$.

<u>An ovoid</u> $\mathcal{O} \Leftrightarrow$ an <u>independent set of vertices</u> in the collinearity graph Γ_2

# of points in $\mathcal{O}$ = $(st+1)$. Count (P, ℓ), $P \in \mathcal{O}$, $P \in \ell$ in two ways

Theorem (Tits ovoids)

The set of points

$$\{\langle(0,1,0,0)\rangle\} \cup \{\langle(1,\ x_3x_4 + x_3^{\sigma} + x_4^{\sigma+2},\ x_3,\ x_4)\rangle \mid x_3, x_4 \in \mathbb{F}_q\}$$

is an ovoid of $W_3(\mathbb{F}_q)$, when q is an odd power of two, and σ is an automorphism of $\mathbb{F}_q$ such that $a^{\sigma^2} = a^2$ for $\forall a \in \mathbb{F}_q$.