Lecture 3. Dependent Random Choice

Theorem 3.1. *Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most r. Then there exists a constant $C = C_H$ such that*

$$ex(n, H) \le Cn^{2-1/r}$$

Remark 3.2. This theorem was first proved by Füredi (1991) and then was reproved by Alon, Krivelevich and Sudakov (2002).

We will give the proof of Alon-Krivelevich-Sudakov, which has been extended to a powerful probabilistic tool called "dependent random choice". The main idea of this is the following lemma: If G has many many edges, then one can find a large subset A in G such that all small subsets of A have many common neighbors.

Definition 3.3. For $S \subseteq V(G)$, $N(S) = \{w \in V(G) : ws \in E(G) \text{ for every } s \in S\}$.

Lemma 3.4 (Dependent random choice)**.** *Let $u, n, r, m, t \in \mathbb{N}$ and a real number $\alpha \in (0, 1)$ be such that*

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \ge u$$

Then every n-vertex graph G with at least $\frac{\alpha}{2}n^2$ edges contains a subset U of at least u vertices such that every r-element subset S of U has at least m common neighbors.

Proof. Let T be a set of t vertices chosen uniformly at random from $V(G)$ (allowing repetition). Let $A = N(T)$. Then

$$\mathbb{E}[|A|] = \sum_{v\in V} \mathbb{P}[v \in A] = \sum_{v\in V} \mathbb{P}[T \subseteq N(v)] = \sum_{v\in V} \left(\frac{d(v)}{n}\right)^t \ge n\left(\frac{1}{n}\sum_{v\in V}\frac{d(v)}{n}\right)^t \ge n\alpha^t.$$

Call an r-element subset $S \subseteq V(G)$ *bad* if $|N(S)| < m$. Given a bad r-set $S \subseteq V(G)$, we have

$$\mathbb{P}[S \subseteq A] = \mathbb{P}[T \subseteq N(S)] = \left(\frac{|N(S)|}{n}\right)^t < \left(\frac{m}{n}\right)^t.$$

Let s be the number of bad r-subsets in A, so

$$\mathbb{E}[s] < \binom{n}{r}\left(\frac{m}{n}\right)^t,$$

$$\mathbb{E}[|A| - s] \ge n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \ge u.$$

Thus, there exists a choice of T such that $A = N(T)$ satisfies that $|A| - s \ge u$. Let U be obtained from A by deleting one vertex from each bad r-element subset in A. Then we have that $|U| \ge u$ and U satisfies the condition. ∎

Now we can prove the Theorem 3.1.

Proof. (Theorem 3.1) Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most r. We want to show $\text{ex}(n, H) \leq Cn^{2-1/r}$, where $C = C_H$ is a constant. Let G be any n-vertex graph with at least $Cn^{2-1/r}$ edges, where C satisfies

$$n(2Cn^{-1/r})^r - \binom{n}{r}\left(\frac{|A| + |B|}{n}\right)^r \geq |B|.$$

By Lemma 3.4, taking $u = |B|$, $m = |A| + |B|$, $t = r$, $\alpha = 2Cn^{-1/r}$, we see

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u.$$

So there exists a subset U with $|U| \geq u$ such that any r-element subsets of U has at least $m = |A| + |B|$ common neighbors.

We label $A = \{v_1, v_2, ..., v_a\}$ and $B = \{u_1, u_2, ..., u_b\}$. We find any one-to-one mapping $\phi : B \to U$, $u_i \mapsto \phi(u_i)$. Next, we want to extend this ϕ from B to $A \cup B$ and then we can find a copy of H in G. Suppose for $A' = \{v_1, v_2, ..., v_s\}$, we have $\phi : A' \cup B \to V(G)$ such that $H[A' \cup B] \subseteq G[\phi(A') \cup \phi(B')]$. Consider v_{s+1} and $N_H(v_{s+1}) \subseteq B$, we have that $N_H(v_{s+1}) \leq r$. We consider $\phi(N_H(v_{s+1})) \subseteq U$ of size at most r. By the property of U, $\phi(N_H(v_{s+1}))$ has at least $|A|+|B|$ common neighbors in G. Then we can get a vertex $\phi(v_{s+1})$ which is a common neighbor of $\phi(N_H(v_{s+1}))$ but is not in $\phi(A' \cup B)$. Repeatedly, we can extend ϕ to be $\phi : A \cup B \to V(G)$ such that $\phi(A \cup B)$ is a copy of H, a contradiction.

∎

A *subdivision* of a graph H is obtained from H by replacing each edge xy in H with a path $xP_{xy}y$ such that all P_{xy}s are distinct.

Theorem 3.5. *Any n-vertex graph G with at least εn^2 edges has a subdivision of a clique of size at least $\varepsilon^{3/2}n^{1/2}$.*

Proof. This is left to be an exercise. ∎

Lemma 3.6 (Two-sided version of dependent random choice)**.** *Let G be a bipartite graph on $2n$ vertices and with average degree d. Let U, V be two parts of G with $|U| = |V| = n$. If $r, s, t \in \mathbb{N}$ such that*

$$n^{r-s+s^2}d^{-s^2}(t-1)^s < \frac{1}{4}.$$

Then there exist $X \subseteq U$ and $Y \subseteq V$ of size at least $4^{-1/s}n^{1-s}d^s$ satisfying that every r-subset in X (or in Y) has a least t common neighbors in $G(X, Y)$.

A graph H is r-*degenerate* if any one of its subgraphs contains a vertex of degree at most r.

Theorem 3.7. *Let $r \geq 2$ and F be an r-degenerate bipartite graph whose largest part has size t. Then there exists a constant $C = C(F)$ such that*

$$\text{ex}(n, F) \leq C(t-1)^{\frac{1}{2r}}n^{2-\frac{1}{4r}}.$$

Proof. Let C be the constant such that $(\frac{C}{2})^{-4r^2} < \frac{1}{4}$. Let G be a $(2n)$-vertex graph with $e(G) > C(t-1)^{\frac{1}{2r}} n^{2-\frac{1}{4r}}$. Thus, its average degree $d > \frac{C}{2}(t-1)^{\frac{1}{2r}} n^{1-\frac{1}{4r}}$. We know that there exists a subgraph G' of G, which is bipartite with parts U, V of size n and $e(G') \geq e(G)/2$. Let $s = 2r$. It is easy to see that

$$n^{r-s+s^2} d^{-s^2} (t-1)^s < \frac{1}{4},$$

since the choice of C and the inequality implies $4^{-1/s} n^{1-s} d^s \geq t$. By Lemma 3.6, we obtain that there exist $X \subseteq U$ and $Y \subseteq V$ of size at least $4^{-1/s} n^{1-s} d^s$ satisfying that every r-subset in X(or in Y) has a least t common neighbors in $G(X, Y)$.

Let F be a bipartite graph on partition $A \cup B$. Our goal is to construct an embedding $f : V(F) \to V(G)$ by placing images of vertices from A into X, and images of vertices of B into Y. To construct the desired embedding, we proceed according to the chosen order $(v_1, \ldots, v_h)$ of the vertices of F. If the current vertex $v_i \in V(F), i \in [h]$ is a vertex from A, we first locate the images $f(v_j)$, $j < i$, of the already embedded neighbours of v_i in B. The set $\{f(v_j) : j < i, (v_j, v_i) \in E(H)\}$ is a subset of Y of cardinality at most r. It therefore has at least t common neighbours in X, and obviously not all of them have already been used in the embedding. We pick one unused vertex w and set $f(v_i) = w$. If $v_i \in B$, we can repeat the above argument, interchanging the roles of X and Y. We can find a copy of F in (X, Y), a contradiction. So, we have

$$\mathrm{ex}(n, F) \leq C(t-1)^{\frac{1}{2r}} n^{2-\frac{1}{4r}}.$$

∎

Corollary 3.8. *For any bipartite graph F, let $d_F = \max_{F' \subseteq F} \frac{2e(F')}{v(F')}$. Then*

$$\mathrm{ex}(n, F) = O(n^{2-\frac{1}{4\lfloor d_F \rfloor}}) = O(n^{2-\frac{1}{4d_F}}).$$

Hint: It holds since F is $\lfloor d_F \rfloor$-degenerate.

Corollary 3.9. *For bipartite graph F, let*

$$c_F = \min_{F' \subseteq F} \frac{v(F')}{e(F')}$$

and

$$c_F^* = \min_{F' \subseteq F, e(F') \geq 2, \delta(F') \geq 1} \frac{v(F') - 2}{e(F') - 1}.$$

Then

$$\mathrm{ex}(n, F) = \Omega(n^{2-c_F^*}) \geq \Omega(n^{2-c_F}).$$