Lecture 2. Random Algebraic Constructions

In this lecture, we use random algebraic/polynomial construction to prove the following result, which gives a weaker bound than Theorem 1.10.

Theorem 2.1. *For any s, there exists $C = C(s)$ such that for any $t \geq C$, $\mathrm{ex}(n, K_{s,t}) = \Omega_{s,t}(n^{2-\frac{1}{s}})$.*

Proof. Let q be a prime power, and F_q be the field of order q. Let $s \geq 4$ be fixed and $q \gg s$. Let $d = s^2 - s + 2$, and $n = q^s$.

Definition 2.2. Let $\vec{X} = \{x_1, x_2, ..., x_s\} \in F_q^s$ and $\vec{Y} = \{y_1, y_2, ..., y_s\} \in F_q^s$. Let $\mathcal{P}$ be all polynomials $f(\vec{X}, \vec{Y})$ of degree at most d in each of $\vec{X}$ and $\vec{Y}$, that is,

$$f(\vec{X}, \vec{Y}) = \sum_{(\vec{a}, \vec{b})} \alpha_{\vec{a},\vec{b}} \cdot x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \cdot y_1^{b_1} y_2^{b_2} \cdots y_s^{b_s},$$

over all possible choices that $\sum_{i \in [s]} a_i \leq d$ and $\sum_{j \in [s]} b_j \leq d$, where $\alpha_{\vec{a},\vec{b}} \in F_q$.

We choose a polynomial $f \in \mathcal{P}$ randomly at uniform and use it to define a bipartite graph G_f on partition (F_q^s, F_q^s) with edge set $\{(\vec{X}, \vec{Y}) : f(\vec{X}, \vec{Y}) = 0\}$. Note that $v(G_f) = 2q^s = 2n$. Then by the linearity of expectation, $\mathbb{E}[e(G_f)] = n^2/q = n^{2-1/s}$.

Lemma 2.3. *For any $\vec{u}, \vec{v} \in F_q^s$, $\mathbb{P}[f(\vec{u}, \vec{v}) = 0] = 1/q$.*

Lemma 2.4. *Suppose $r, s \leq \min\{\sqrt{q}, d\}$. Let $U \subseteq F_q^s$ and $V \subseteq F_q^s$ be sets with $|U| = s$ and $|V| = r$. Then*

$$\mathbb{P}[f(\vec{u}, \vec{v}) = 0 \textit{ for all } \vec{u} \in U, \textit{ and } \vec{v} \in V] = 1/q^{sr}.$$

Fix $U \subseteq F_q^s$ with $|U| = s$. Let $I(\vec{v}) = 1$ if $\vec{v}$ is adjacent to any $\vec{u} \in U$, and otherwise $I(\vec{v}) = 0$. Let $X_U = |N(U)|$. Then $X_U = \sum_{\vec{v}} I(\vec{v})$. We have

$$\begin{aligned}\mathbb{E}[X_U^d] = \mathbb{E}[(\sum_{\vec{v} \in F_q^s} I(\vec{v}))^d] = \sum_{\vec{v_1}, \cdots, \vec{v_d} \in F_q^s} \mathbb{E}[I(\vec{v_1}) I(\vec{v_2}) \cdots I(\vec{v_d})] = \sum_{1 \leq r \leq d} \binom{q^s}{r} \frac{1}{q^{rs}} M_r \\ \leq \sum_{r \leq d} M_r \triangleq M,\end{aligned}$$

where M_r is defined to the number of surjective mappings from $[d]$ to $[r]$.

Lemma 2.5. *For all s, d, there exists a constant C such that if $f_1(\vec{Y}), f_2(\vec{Y}), ..., f_s(\vec{Y})$ are polynomials over $Y \in F_q^s$ of degree at most d, then*

$$\{\vec{y} \in F_q^s : f_1(\vec{y}) = f_2(\vec{y}) = ... = f_s(\vec{y}) = 0\}$$

has size either at most C or at least $q - C\sqrt{q} \geq q/2$.

By lemma 2.5, if $X_U > C$, then $X_U > q/2$ implies

$$\mathbb{P}[X_U > C] = \mathbb{P}[X_U \geq \frac{q}{2}] = \mathbb{P}[X_U^d \geq (\frac{q}{2})^d] \leq \frac{\mathbb{E}[X_U^d]}{(q/2)^d} \leq \frac{M}{(q/2)^d}.$$

We say a set U of s vertices is *bad* if $X_U > C$. Let u be the number of bad sets U of size s. So we have $\mathbb{E}[u] \le \binom{q^s}{s}\frac{M}{(q/2)^d} = O(q^{s-2})$ and $\mathbb{E}[e(G_f) - nu] \ge \frac{n^2}{q} - nO(q^{s-2}) \ge \frac{n^2}{2q} = \frac{1}{2}n^{2-1/s}$. Take such a G_f and remove a vertex from every such s-subset to create a new graph G'. We see that G' is $K_{s,C+1}$-free, $v(G') \le 2n$, and

$$e(G') \ge e(G) - u \cdot n \ge \frac{n^2}{q} - O(q^{s-2})n = (1 - o(1))n^{2-\frac{1}{s}}.$$

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Theorem 2.6 (Bukh-Conlon)**.** *For any rational number $r \in (1,2)$, there is a family of graphs $\mathcal{F}_r$ such that* $\mathrm{ex}(n, \mathcal{F}_r) = \Theta(n^r)$.

Given a rooted tree T with a set R of roots, then p^{th} power $\mathcal{T}^p$ of T is the family of graphs consisting of all possible unions of p distinct labelled copies of T, each of which agree on R.

Definition 2.7. The *density* of a rooted tree (T, R) is defined by

$$\rho_T = \frac{e(T)}{v(T) - |R|}.$$

For any $S \subseteq V(T) \setminus R$, define

$$\rho_S = \frac{\text{The number of edges incident to } S}{|S|}.$$

A rooted tree (T, R) is *balanced* if for any $S \subseteq V(T) \setminus R$, $\rho_S \ge \rho_T$.

Theorem 2.8 (Bukh-Conlon)**.** *For large p,* $\mathrm{ex}(n, \mathcal{T}^p) = \Theta(n^{2-1/\rho_T})$.