Lecture 4. Lower bounds on Ramsey numbers

4.1 Lovász Local Lemma

Theorem 4.1 (Lovász Local Lemma (Symmetric Version))**.** *Let $\{A_i\}_{i\in[k]}$ be a family of random events. For any i, $\mathbb{P}[A_i] \leq p$. Any event is independent of all other events except for d of them. If $ep(d-1) < 1$, then the probability that all events' complements occur simultaneously is greater than 0, that is:*

$$\mathbb{P}\left[\bigcap A_i^c\right] > 0.$$

We focus on the asymmetric version of the Lovász Local Lemma, which is stronger than the symmetric version. Firstly, we define the following auxiliary graph.

Definition 4.2. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a family of random events in the probability space Ω. Let $D := D_{\mathcal{A}}$ be the *dependence graph* with vertex set $V(D) = \mathcal{A}$ and edge set $E(D) = \{A_iA_j :$ A_i and A_j are dependent for each $i, j \in [k]\}$.

Theorem 4.3 (Lovász Local Lemma (Asymmetric Version))**.** *Given a probability space $(\Omega, \mathbb{P})$, the event collection $\mathcal{A}$ and the dependence graph D in Definition 4.2. Denote the neighborhood set of A_i in D by $N(A_i)$. If there exists a mapping $f : \mathcal{A} \to [0, 1)$ satisfying that*

$$\mathbb{P}[A_i] \leq f(A_i) \prod_{B\in N(A_i)} (1 - f(B))$$

holds for each $A_i \in \mathcal{A}$, then the following holds:

$$\mathbb{P}\left[\bigcap A_i^c\right] > 0.$$

4.2 Applications of Lovász Local Lemma

The Ramsey number $r(k, \ell)$ is the smallest integer N such that any red-blue edge-coloring of K_N contains a red K_k or a blue K_ℓ.

Remark:

- Ramsey's Theorem [3]: The Ramsey number exists.
- Erdős and Szekeres [2]: $r(k, \ell) \leq \binom{k+\ell-2}{k-1}$. If $k = \ell$, then this yields $r(k, k) \leq 4^k$.
- Campos, Griffiths, Morris and Sahasrabudhe [1]: There exists $\varepsilon > 0$ such that $r(k, k) \leq (4 - \varepsilon)^k$.

Theorem 4.4.

$$r(3, \ell) = \Omega\left(\frac{\ell^2}{\log^2 \ell}\right).$$

Proof. Consider a random edge-coloring of the complete graph K_n, where each edge is colored red with probability p and blue with probability $1 - p$ independently. Our goal is to obtain that with positive probability there is a coloring without a red triangle and without a blue K_ℓ, since this would establish the lower bound $r(3, \ell) > n$.

For each 3-element set $T \subseteq [n]$, let A_T be the event that T induces a red K_3. Note that for each T, we have $\mathbb{P}[A_T] = p^3$. For each ℓ-element set $S \subseteq [n]$, let B_S be the event that S induces a blue K_ℓ. Note that for each ℓ, we have $\mathbb{P}[B_S] = (1-p)^{\binom{\ell}{2}}$. Let us now define a dependence graph D for these events. We join two events of the form A_T or B_S, if the corresponding sets S or T share an edge. Now we can bound the degrees in this graph,

$$\begin{cases} \deg(A_T) \le \binom{3}{2}(n-3) + \binom{3}{2}\binom{n-3}{\ell-2} \le 3n + \binom{n}{\ell} \text{ for each } A_T, \\ \deg(B_S) \le \binom{\ell}{2}n + \binom{\ell}{2}\binom{n-3}{\ell-2} \le \binom{\ell}{2}n + \binom{n}{\ell} \text{ for each } B_S. \end{cases}$$

Define a mapping $f : \{A_T : T \in \binom{[n]}{3}\} \cup \{B_S : S \in \binom{[n]}{\ell}\} \to [0,1)$ such that $x := f(A_T)$ and $y := f(B_S)$. In order to use Lovász Local Lemma, we need to find positive real numbers $x, y \in (0,1)$ such that

$$\begin{cases} p^3 \le x(1-x)^{3n}(1-y)^{\binom{n}{\ell}}, \\ (1-p)^{\binom{\ell}{2}} \le y(1-x)^{n\binom{\ell}{2}}(1-y)^{\binom{n}{\ell}}, \end{cases}$$

Let us now try to find such $p, x, y \in (0,1)$ for sufficiently large n. We choose $y = \frac{1}{\binom{n}{\ell}}$, then $(1-y)^{\binom{n}{\ell}} \approx \frac{1}{e}$. Furthermore, we observe that p and x need to fulfill the following inequalities:

$$p^3 \le x(1-x)^{3n}(1-y)^{n/2} \le x,$$

$$e^{-p\binom{\ell}{2}} \approx (1-p)^{\binom{\ell}{2}} \le y(1-x)^{\binom{\ell}{2}n}(1-y)^{n/2} \le (1-x)^{\binom{\ell}{2}n} \approx e^{-xn\binom{\ell}{2}}.$$

Hence we need $p \ge xn \ge p^3 n$. Therefore $p \le \frac{1}{\sqrt{n}}$ and $x \ge p^3$. Finally, for the second condition, we note

$$e^{-p\binom{\ell}{2}} \approx (1-p)^{\binom{\ell}{2}} \le y(1-x)^{\binom{\ell}{2}n}(1-y)^{n/2} \le y = \frac{1}{\binom{n}{\ell}} \approx e^{-\ell \log n},$$

hence $p\ell^2 \ge p\binom{\ell}{2} \ge \ell \log n$ and therefore $\ell \ge \frac{1}{p}\log n \ge \sqrt{n}\log n$.

Motivated by this we may assume $\ell \ge 20\sqrt{n}\log n$ and choose $y = \frac{1}{\binom{n}{\ell}}, x = \frac{1}{9n^{3/2}}$ and $p = \frac{1}{3\sqrt{n}}$. After choosing the constants, we can give the full proof. For sufficiently large n, we have

$$(1-y)^{\binom{n}{\ell}} = \left(1 - \frac{1}{\binom{n}{\ell}}\right)^{\binom{n}{\ell}} \ge e^{-1.01},$$

$$(1-x)^{3n} = \left(1 - \frac{1}{9n^{3/2}}\right)^{3n} \ge 1 - \frac{1}{3\sqrt{n}} \ge e^{-0.01}.$$

Thus,

$$p^3 = \frac{1}{27n^{3/2}} \le \frac{1}{9n^{3/2}} \cdot \frac{1}{3} \le \frac{1}{9n^{3/2}}e^{-1.02} \le x(1-x)^{3n}(1-y)^{\binom{n}{\ell}},$$

which establishes the first desired inequality.

For the second inequality, for sufficiently large n, we get

$$(1-x)^{\binom{l}{2}n} \ge e^{-2xn\binom{l}{2}} \ge e^{-\frac{2}{9\sqrt{n}}\binom{\ell}{2}}.$$

Furthermore, using $\ell \ge 20\sqrt{n}\log n$, we have

$$y = \frac{1}{\binom{n}{\ell}} \ge \frac{1}{n^\ell} = e^{-l\log n} \ge e^{-l^2\frac{1}{20\sqrt{n}}} \ge e^{-\ell(\ell-1)\frac{1}{19\sqrt{n}}} \ge e^{-\frac{1}{9\sqrt{n}}\binom{\ell}{2}+1.01}.$$

Hence

$$(1-p)^{\binom{\ell}{2}} \le e^{-p\binom{\ell}{2}} = e^{-\frac{1}{3\sqrt{n}}\binom{\ell}{2}} = e^{-\frac{1}{9\sqrt{n}}\binom{\ell}{2}+1.01} e^{-\frac{2}{9\sqrt{n}}\binom{\ell}{2}} e^{-1.01} \le y(1-x)^{\binom{\ell}{2}n}(1-y)^{\binom{n}{\ell}},$$

which verifies the second desired inequality.

By Lemma 4.3, we obtain that

$$\mathbb{P}\left[\bigcap_{T\in\binom{[n]}{3}} A_T^c \cap \bigcap_{S\in\binom{[n]}{\ell}} B_S^c\right] > 0.$$

So for $\ell \ge 20\sqrt{n}\log n$, we can find p, x, y such that there exists a 2 edge-coloring of K_n such that there is no red triangle and there is no blue K_ℓ. This implies $r(3,\ell) > n$.

Note that $n \le \frac{\ell^2}{(40\log\ell)^2}$ implies

$$20\sqrt{n}\log n \le 20\frac{\ell}{40\log\ell}\log\ell^2 = \ell.$$

Therefore we have $r(3,\ell) \ge \frac{\ell^2}{(40\log\ell)^2}$. ∎

Theorem 4.5 (Erdős' Lower Bound). *Let $C \ge 1$ and $p_C \in (0,1/2]$ be the unique solution to $C = \frac{\log p_C}{\log(1-p_C)}$. Let $M_C = p_C^{-1/2}$. Then $r(\ell, C\ell) = \Omega(\ell\cdot M_C^\ell)$. In particular, when $C = 1$, we have $p_C = 1$ and $r(\ell,\ell) = \Omega(\ell\sqrt{2}^\ell)$.*

Proof. Let $p \in (0,1/2]$. Consider a random edge-coloring of the complete graph K_n, where each edge is independently colored red with probability p and blue with probability $1-p$. Let

$$f(n,p) := A(n,p) + B(n,p) \text{ where } A(n,p) = \binom{n}{\ell}p^{\binom{\ell}{2}} \text{ and } B(n,p) = \binom{n}{C\ell}(1-p)^{\binom{C\ell}{2}}.$$

Note that

$$\mathbb{P}[\text{There exists a red } K_\ell \text{ or a blue } K_{C\ell}] \le f(n,p).$$

Hence, if $f(n,p) = 1 - o_\ell(1)$, then there exists at least one such coloring with no red K_ℓ and no blue $K_{C\ell}$, implying $r(\ell, C\ell) > n$. It thus suffices to find the maximum value of $n = n(p)$ such that $f(n,p) = o_\ell(1)$. Assume this maximum is achieved at $p_0 = p_{C,\ell}$. Then,

$$\frac{\partial f(n,p_0)}{\partial p} = \frac{\binom{\ell}{2}}{p_0}\binom{n}{\ell}p_0^{\binom{\ell}{2}} - \frac{\binom{C\ell}{2}}{1-p_0}\binom{n}{C\ell}(1-p_0)^{\binom{C\ell}{2}} = 0.$$

Thus, we have $\log A(n,p_0) = \log B(n,p_0) + O(\log\ell)$. Solving this along with $A(n,p_0)+B(n,p_0) = 1-o_\ell(1)$, we obtain that $\log A(n,p_0) = O(\log\ell)$ and $\log B(n,p_0) = O(\log\ell)$. Therefore, we obtain that

$$\begin{cases} -\log p_0 = \frac{2\log(en/\ell)}{\ell-1} + O\left(\frac{\log\ell}{\ell^2}\right), \\ -\log(1-p_0) = \frac{2\log(en/C\ell)}{C\ell-1} + O\left(\frac{\log\ell}{\ell^2}\right). \end{cases}$$

We then derive that $p_0 = p_C + O(1/\ell)$, where the constant p_C satisfies $C = \frac{\log p_C}{\log(1-p_C)}$. It follows directly from the above that $n = \frac{\ell}{e}\cdot p_0^{-(\ell-1)/2}\cdot e^{O\left(\frac{\log\ell}{\ell}\right)} = \Theta(\ell)\cdot(p_C + O(1/\ell))^{-\ell/2} = \Theta(\ell\cdot M_C^\ell)$, where $M_C := p_C^{-1/2}$. This establishes $r(\ell, C\ell) = \Omega(\ell\cdot M_C^\ell)$. ∎

References

[1] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe. An exponential improvement for diagonal Ramsey. *arXiv:2303.09521*, 2025.

[2] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compos. Math.*, 2:463–470, 1935.

[3] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930.

Extensive reading

The following two papers provide relevant extensive reading materials:

1. Noga Alon, Lajos Rónyai and Tibor Szabó, Norm-graphs: variations and applications, Journal of Combinatorial Theory, Series B 76 (1999), 280–290.
2. B. Bukh and D. Conlon, Rational exponents in extremal graph theory, Journal of the European Mathematical Society, 20 (2018), 1747–1757.