Constructive Proofs in Extremal Graph Theory

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Lecture 1. Constructions of Extremal Graphs

Given a graph H, a graph G is called *H-free* if G does not contain H as its subgraph. For general graphs H, the *Turán number* $\text{ex}(n, H)$ is defined as follows:

$$\text{ex}(n, H) = \max\{e(G) : v(G) = n, H \not\subseteq G\}.$$

Kövari-Sós-Turán Theorem tells us that for any bipartite H there exists $c > 0$ such that $\text{ex}(n, H) = O(n^{2-c})$. Now we will apply Randomized construction, Algebraic construction, and Randomized Algebraic construction to obtain lower bounds of Turán numbers for bipartite graphs.

1.1 Randomized construction

Theorem 1.1. *For any graph H with at least 2 edges, there exists a constant $c > 0$ such that*

$$\text{ex}(n, H) \geq cn^{2-\frac{v(H)-2}{e(H)-1}}.$$

Proof. (The idea is to use random graphs and the deletion/alternation method.) Consider a random graph $G = G(n, p)$ where $p = \frac{1}{2}n^{\frac{v(H)-2}{e(H)-1}}$. Let h be the number of H-copies in G. Then we have

$$\mathbb{E}[h] = \frac{n(n-1)\cdots(n-v(H)+1)}{|Aut(H)|}p^{e(H)} \leq n^{v(H)}p^{e(H)}.$$

Since $p = \frac{1}{2}n^{-\frac{v(H)-2}{e(H)-1}}$ and $\mathbb{E}[e(G)] = p\binom{n}{2}$, we get $\mathbb{E}[h] \leq \mathbb{E}[e(G)]/2$ which implies that

$$\mathbb{E}[e(G) - h] \geq \frac{1}{2}\mathbb{E}[e(G)] = \frac{1}{2}p\binom{n}{2} \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

Thus there exists an n-vertex graph G with $e(G) - h \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}$.

Let G' be obtained from G by deleting one edge for each copy of H in G. Then G' is H-free and

$$e(G') \geq e(G) - h \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

So

$$\text{ex}(n, H) \geq e(G') \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

∎

Remark:

- $\mathrm{ex}(n, C_{2k}) = \Omega(n^{2-\frac{2k-2}{2k-1}}) = \Omega(n^{1+\frac{1}{2k-1}})$.
- $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{s+t-2}{st-1}})$.

Definition 1.2. The *2-density* of H is

$$m_2(H) = \max_{\substack{H' \subset H \\ v(H') \geq 3}} \frac{e(H') - 1}{v(H') - 2}.$$

Exercise:

- For any H with at least 2 edges,

$$\mathrm{ex}(n, H) = \Omega(n^{2-\frac{1}{m_2(H)}}).$$

1.2 Algebraic construction

For C_4, we have Reiman's bound:

$$\mathrm{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) = (\frac{1}{2} + o(1))n^{\frac{3}{2}}.$$

Next, we will give the lower bound of $\mathrm{ex}(n, C_4)$ using algebraic construction. We can prove the following theorem.

Theorem 1.3. $\mathrm{ex}(n, C_4) \geq (\frac{1}{2} + o(1))n^{3/2}$.

Proof. For a prime q, we first define the Erdős-Rényi polarity graph ER_q as following:

- Its vertex set is $\{U : U$ is a 1-dimension subspace in a 3-dimension space $\mathbb{F}_q^3\}$.
- U, W are adjacent in ER_q if and only if U and W ($U \neq W$) are perpendicular as 1-dimension subspace.

Obviously $v(ER_q) = \frac{q^3-1}{q-1} = q^2 + q + 1$.

We see each vertex U has degree q or $q+1$, since there are exactly $\frac{q^2-1}{q-1} = q+1$ many 1-dimension subspaces W that are perpendicular to U. But there are $q+1$ absolute vertices U, which means $U \perp U$ and we do not allow loops. For such U, it has degree q. Also ER_q is C_4-free, because given any two vertices U, W, there is exactly one line L perpendicular to both U and W. Then we have

$$e(ER_q) \geq \frac{1}{2}(q^2(q+1) + (q+1)q) = \frac{1}{2}q(q+1)^2 = (\frac{1}{2} + o(1))v(ER_q)^{3/2},$$

where $v(ER_q) = q^2+q+1$ for primes q. By the number theory we know that for any large integer n there exists a prime in the interval $[n - n^{0.525}, n]$. Thus there exists an n-vertex C_4-free graph with at least $(\frac{1}{2} + o(1))n^{3/2}$ edges for any large n. ∎

1.2.1 New constructions of $\text{ex}(n, C_4)$

Theorem 1.4 (Erdős-Rényi-Sós)**.** $\text{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$.

Proof. We have seen that the Erdős-Rényi polarity graphs ER_q can give this lower bound in Theorem 1.3. Now we give a different construction for C_4-free graphs which also yield the same lower bound.

Suppose $n = q^2 - 1$ for a prime q. Consider the following graph $G = (V, E)$, where $V = F_q^2 \setminus \{0, 0\}$, $E = \{(x, y) \sim (a, b) | ax + by = 1 \text{ over } F_q\}$. First, we see G is C_4-free: for any distinct vertices $(a, b) \neq (a', b')$, there is at most one solution(common neighbor) satisfying both $ax + by = 1$ and $a'x + b'y = 1$. It is easy to see that the degree of each vertex is q or $q - 1$ since we do not allow loops. So $|E| \geq \frac{1}{2}(q^2 - 1)(q - 1) \approx (\frac{1}{2} - o(1))n^{3/2}$ (where $n = q^2 - 1$). The above construction works when n is prime. But it known for every integer n there exists a prime p satisfying $n \leq p \leq (1 + o(1))n$, the above lower bound applies to all values of n. We can get $\text{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$. ∎

Remark:

- Bondy-Simonovits: $\text{ex}(n, C_{2k}) \leq 100kn^{1+1/k}$,
- $0.538n^{4/3} \leq \text{ex}(n, C_6) \leq 0.627n^{4/3}$,
- $\text{ex}(n, C_{10}) = \Theta(n^{6/5})$,
- For any $t \geq s \geq 2$, we have $\text{ex}(n, K_{s,t}) = O_{s,t}(n^{2-1/s})$.

1.2.2 Constructions of $\text{ex}(n, K_{3,3})$

Theorem 1.5 (Brown)**.**

$$\text{ex}(n, K_{3,3}) \geq (\frac{1}{2} - o(1))n^{5/3}.$$

Proof. Let $n = q^3$ for some odd prime q. Consider the following graph G with $V(G) = F_q^3$ and $E(G) = \{(x, y, z) \sim (a, b, c) | (x - a)^2 + (y - b)^2 + (z - c)^2 = d \text{ over } F_q\}$, where $d \neq 0$ is a quadratic residue[1] if $q = 4k - 1$ and d is a quadratic non-residue if $q = 4k - 3$.

It is easy to check that G is $K_{3,3}$-free. We should omit the detailed proof, instead we give the following intuition : The $K_{3,3}$-freeness is equivalent to the statement that any 3 unit spheres have at most two common points. It is not hard to see that vertices (x, y, z) have q^2 or $q^2 - 1$ neighbors. Thus we have $e(G) \geq \frac{1}{2}q^3(q^2 - 1) \approx (\frac{1}{2} - o(1))n^{5/3}$ when $n = q^3$. ∎

1.3 Norm grahs

Lemma 1.6. *Let K be a field and $a_{ij}, b_i \in K$ for $1 \leq i, j \leq 2$ such that $a_{1j} \neq a_{2j}$. Then the system of equations*

$$\begin{cases} (x_1 - a_{11})(x_2 - a_{12}) = b_1 \\ (x_1 - a_{21})(x_2 - a_{22}) = b_2 \end{cases}$$

has at most two solutions $(x_1, x_2) \in K \times K$.

[1] there is an integer x such that $x^2 \equiv d \pmod q$.

Proof. Considering the difference of two equations, we get $(a_{11}-a_{21})x_2+(a_{12}-a_{22})x_1+a_{21}a_{22}-a_{11}a_{12}=b_2-b_1$. Since $a_{11}-a_{21}\neq 0$, we can express x_1 by an expression of x_2. Substituting this expression to any one of the equation, we get a quadratic equation in the variable x_2. It has at most two solutions for x_2, each of which determines the valus of x_1. So we have at most two solutions $(x_1,x_2)\in K\times K$. ∎

Lemma 1.7. *Let K be a field with characteristic q. Then any $x,y\in K$ satisfy $(x+y)^q=x^q+y^q$.*

Definition 1.8. Let q be a prime. The *norm mapping* $N: F_{q^s}\to F_q$ is given by

$$N(x)=xx^qx^{q^2}\dots x^{q^{s-1}}$$

for any $x\in F_{q^s}$.

Note that this is well-defined: since $x^{q^s}=x$ for any $x\in F_{q^s}$, we have $(N(x))^q=x^qx^{q^2}\dots x^{q^s}=N(x)$, implying that $N(x)\in F_q$.

Theorem 1.9 (Alon-Rónyai-Szabó). *For every $n=q^3-q^2$ where q is a prime power,*

$$\mathrm{ex}(n,K_{3,3})\geq \frac{1}{2}n^{5/3}+\frac{1}{3}n^{4/3}+C.$$

Proof. Let $N: F_{q^2}\to F_q$ be the norm mapping. The graph $H=H(q,3)$ is as follows. The vertex set of H is $F_{q^2}\times F_q^*$ and $|V(H)|=q^2(q-1)$. Two vertices (A,a) and (B,b) in $V(H)$ are adjacent if and only if $N(A+B)=ab$. The degree of each vertex $(A,a)\in V(H)$ is the number of pairs (B,b) with $N(A+B)=ab$. For any $B\neq -A$, we can have a unique b. So the degree of (A,a) is q^2-1 or q^2-2, as $N(A+A)=a^2$ may happen. So we have $|E(H)|\geq\frac{1}{2}\left(q^2(q-1)\right)(q^2-2)\geq \frac{1}{2}n^{5/3}+\frac{1}{3}n^{4/3}+C$.

Now it suffices to show H is $K_{3,3}$-free, which is enough to show that for any three distinct vertices (D_i,d_i) with $i\in[3]$, they have at most 2 common neighbors. That is, the system of equations:

$$\begin{cases} N(X+D_1)=xd_1 & (1.1)\\ N(X+D_2)=xd_2 & (1.2)\\ N(X+D_3)=xd_3 & (1.3)\end{cases}$$

has at most 2 solutions $(X,x)\in F_{q^2}\times F_q^*$. Observe that if (X,x) is a solution, then:

1) $X\neq -D_i$, for $i\in[3]$, and

2) $D_i\neq D_j$, for $i\neq j$.

Divide equations (1.1) and (1.2) by equation (1.3), we can get

$$\frac{d_i}{d_3}=\frac{N(X+D_i)}{N(X+D_3)}=N\left(\frac{X+D_i}{X+D_3}\right)=N\left(1+\frac{D_i-D_3}{X+D_3}\right),\ \text{for } i=1,2.$$

Let $Y=\frac{1}{X+D_3}, A_i=\frac{1}{D_i-D_3}$, and $b_i=\frac{d_i}{d_3N(D_i-D_3)}, i\in[2]$. Then,

$$\begin{cases}(Y+A_1)(Y^q+A_1^q)=N(Y+A_1)=b_1\\ (Y+A_2)(Y^q+A_2^q)=N(Y+A_2)=b_2\end{cases}$$

It is clear that $A_1\neq A_2$ and $A_1^q\neq A_2^q$. Then by lemma 1.6, this system has at most 2 solutions (Y,Y^q). Therefore, we have at most two pairs of (X,x). ∎

Theorem 1.10 (Alon-Rónyai-Szabó). *$H(q, s)$ is $K_{s,(s-1)!+1}$-free. Therefore, for $t \geq (s-1)!+1$,*

$$\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s}).$$

Proof. Exercise (similar to the proof of Theorem 1.7 for $H(q, 3)$). ∎