Constructive Proofs in Extremal Graph Theory

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Lecture 1. Constructions of Extremal Graphs

Given a graph H, a graph G is called *H-free* if G does not contain H as its subgraph. For general graphs H, the *Turán number* $\mathrm{ex}(n, H)$ is defined as follows:

$$\mathrm{ex}(n, H) = \max\{e(G) : v(G) = n, H \nsubseteq G\}.$$

Kövari-Sós-Turán Theorem tells us that for any bipartite H there exists $c > 0$ such that $\mathrm{ex}(n, H) = O(n^{2-c})$. Now we will apply Randomized construction, Algebraic construction, and Randomized Algebraic construction to obtain lower bounds of Turán numbers for bipartite graphs.

1.1 Randomized construction

Theorem 1.1. *For any graph H with at least 2 edges, there exists a constant $c > 0$ such that*

$$\mathrm{ex}(n, H) \geq cn^{2-\frac{v(H)-2}{e(H)-1}}.$$

Proof. (The idea is to use random graphs and the deletion/alternation method.) Consider a random graph $G = G(n, p)$ where $p = \frac{1}{2}n^{\frac{v(H)-2}{e(H)-1}}$. Let h be the number of H-copies in G. Then we have

$$\mathbb{E}[h] = \frac{n(n-1)\cdots(n-v(H)+1)}{|Aut(H)|}p^{e(H)} \leq n^{v(H)}p^{e(H)}.$$

Since $p = \frac{1}{2}n^{-\frac{v(H)-2}{e(H)-1}}$ and $\mathbb{E}[e(G)] = p\binom{n}{2}$, we get $\mathbb{E}[h] \leq \mathbb{E}[e(G)]/2$ which implies that

$$\mathbb{E}[e(G) - h] \geq \frac{1}{2}\mathbb{E}[e(G)] = \frac{1}{2}p\binom{n}{2} \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

Thus there exists an n-vertex graph G with $e(G) - h \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}$.

Let G' be obtained from G by deleting one edge for each copy of H in G. Then G' is H-free and

$$e(G') \geq e(G) - h \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

So

$$\mathrm{ex}(n, H) \geq e(G') \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

∎

Remark:

- $\mathrm{ex}(n, C_{2k}) = \Omega(n^{2-\frac{2k-2}{2k-1}}) = \Omega(n^{1+\frac{1}{2k-1}})$.
- $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{s+t-2}{st-1}})$.

Definition 1.2. The *2-density* of H is

$$m_2(H) = \max_{\substack{H' \subset H \\ v(H') \geq 3}} \frac{e(H') - 1}{v(H') - 2}.$$

Exercise:

- For any H with at least 2 edges,

$$\mathrm{ex}(n, H) = \Omega(n^{2-\frac{1}{m_2(H)}}).$$

1.2 Algebraic construction

For C_4, we have Reiman's bound:

$$\mathrm{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) = (\frac{1}{2} + o(1))n^{\frac{3}{2}}.$$

Next, we will give the lower bound of $\mathrm{ex}(n, C_4)$ using algebraic construction. We can prove the following theorem.

Theorem 1.3. $\mathrm{ex}(n, C_4) \geq (\frac{1}{2} + o(1))n^{3/2}$.

Proof. For a prime q, we first define the Erdős-Rényi polarity graph ER_q as following:

- Its vertex set is $\{U : U$ is a 1-dimension subspace in a 3-dimension space $\mathbb{F}_q^3\}$.
- U, W are adjacent in ER_q if and only if U and W $(U \neq W)$ are perpendicular as 1-dimension subspace.

Obviously $v(ER_q) = \frac{q^3-1}{q-1} = q^2 + q + 1$.

We see each vertex U has degree q or $q + 1$, since there are exactly $\frac{q^2-1}{q-1} = q + 1$ many 1-dimension subspaces W that are perpendicular to U. But there are $q + 1$ absolute vertices U, which means $U \perp U$ and we do not allow loops. For such U, it has degree q. Also ER_q is C_4-free, because given any two vertices U, W, there is exactly one line L perpendicular to both U and W. Then we have

$$e(ER_q) \geq \frac{1}{2}(q^2(q+1) + (q+1)q) = \frac{1}{2}q(q+1)^2 = (\frac{1}{2} + o(1))v(ER_q)^{3/2},$$

where $v(ER_q) = q^2 + q + 1$ for primes q. By the number theory we know that for any large integer n there exists a prime in the interval $[n - n^{0.525}, n]$. Thus there exists an n-vertex C_4-free graph with at least $(\frac{1}{2} + o(1))n^{3/2}$ edges for any large n. ∎

1.2.1 New constructions of $\mathrm{ex}(n, C_4)$

Theorem 1.4 (Erdős-Rényi-Sós)**.** $\mathrm{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$.

Proof. We have seen that the Erdős-Rényi polarity graphs ER_q can give this lower bound in Theorem 1.3. Now we give a different construction for C_4-free graphs which also yield the same lower bound.

Suppose $n = q^2 - 1$ for a prime q. Consider the following graph $G = (V, E)$, where $V = F_q^2 \setminus \{0, 0\}$, $E = \{(x, y) \sim (a, b) | ax + by = 1 \text{ over } F_q\}$. First, we see G is C_4-free: for any distinct vertices $(a, b) \neq (a', b')$, there is at most one solution(common neighbor) satisfying both $ax + by = 1$ and $a'x + b'y = 1$. It is easy to see that the degree of each vertex is q or $q - 1$ since we do not allow loops. So $|E| \geq \frac{1}{2}(q^2 - 1)(q - 1) \approx (\frac{1}{2} - o(1))n^{3/2}$ (where $n = q^2 - 1$). The above construction works when n is prime. But it known for every integer n there exists a prime p satisfying $n \leq p \leq (1 + o(1))n$, the above lower bound applies to all values of n. We can get $\mathrm{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$. ∎

Remark:

- Bondy-Simonovits: $\mathrm{ex}(n, C_{2k}) \leq 100kn^{1+1/k}$,
- $0.538n^{4/3} \leq \mathrm{ex}(n, C_6) \leq 0.627n^{4/3}$,
- $\mathrm{ex}(n, C_{10}) = \Theta(n^{6/5})$,
- For any $t \geq s \geq 2$, we have $\mathrm{ex}(n, K_{s,t}) = O_{s,t}(n^{2-1/s})$.

1.2.2 Constructions of $\mathrm{ex}(n, K_{3,3})$

Theorem 1.5 (Brown)**.**

$$\mathrm{ex}(n, K_{3,3}) \geq (\frac{1}{2} - o(1))n^{5/3}.$$

Proof. Let $n = q^3$ for some odd prime q. Consider the following graph G with $V(G) = F_q^3$ and $E(G) = \{(x, y, z) \sim (a, b, c) | (x - a)^2 + (y - b)^2 + (z - c)^2 = d \text{ over } F_q\}$, where $d \neq 0$ is a quadratic residue[1] if $q = 4k - 1$ and d is a quadratic non-residue if $q = 4k - 3$.

It is easy to check that G is $K_{3,3}$-free. We should omit the detailed proof, instead we give the following intuition : The $K_{3,3}$-freeness is equivalent to the statement that any 3 unit spheres have at most two common points. It is not hard to see that vertices (x, y, z) have q^2 or $q^2 - 1$ neighbors. Thus we have $e(G) \geq \frac{1}{2}q^3(q^2 - 1) \approx (\frac{1}{2} - o(1))n^{5/3}$ when $n = q^3$. ∎

1.3 Norm grahs

Lemma 1.6. *Let K be a field and $a_{ij}, b_i \in K$ for $1 \leq i, j \leq 2$ such that $a_{1j} \neq a_{2j}$. Then the system of equations*

$$\begin{cases} (x_1 - a_{11})(x_2 - a_{12}) = b_1 \\ (x_1 - a_{21})(x_2 - a_{22}) = b_2 \end{cases}$$

has at most two solutions $(x_1, x_2) \in K \times K$.

[1] there is an integer x such that $x^2 \equiv d \pmod{q}$.

Proof. Considering the difference of two equations, we get $(a_{11}-a_{21})x_2+(a_{12}-a_{22})x_1+a_{21}a_{22}-a_{11}a_{12}=b_2-b_1$. Since $a_{11}-a_{21}\neq 0$, we can express x_1 by an expression of x_2. Substituting this expression to any one of the equation, we get a quadratic equation in the variable x_2. It has at most two solutions for x_2, each of which determines the valus of x_1. So we have at most two solutions $(x_1,x_2)\in K\times K$. ∎

Lemma 1.7. *Let K be a field with characteristic q. Then any $x,y\in K$ satisfy $(x+y)^q=x^q+y^q$.*

Definition 1.8. Let q be a prime. The *norm mapping* $N: F_{q^s}\to F_q$ is given by

$$N(x)=xx^qx^{q^2}\dots x^{q^{s-1}}$$

for any $x\in F_{q^s}$.

Note that this is well-defined: since $x^{q^s}=x$ for any $x\in F_{q^s}$, we have $(N(x))^q=x^qx^{q^2}\dots x^{q^s}=N(x)$, implying that $N(x)\in F_q$.

Theorem 1.9 (Alon-Rónyai-Szabó)**.** *For every $n=q^3-q^2$ where q is a prime power,*

$$\mathrm{ex}(n,K_{3,3})\geq \frac{1}{2}n^{5/3}+\frac{1}{3}n^{4/3}+C.$$

Proof. Let $N: F_{q^2}\to F_q$ be the norm mapping. The graph $H=H(q,3)$ is as follows. The vertex set of H is $F_{q^2}\times F_q^*$ and $|V(H)|=q^2(q-1)$. Two vertices (A,a) and (B,b) in $V(H)$ are adjacent if and only if $N(A+B)=ab$. The degree of each vertex $(A,a)\in V(H)$ is the number of pairs (B,b) with $N(A+B)=ab$. For any $B\neq -A$, we can have a unique b. So the degree of (A,a) is q^2-1 or q^2-2, as $N(A+A)=a^2$ may happen. So we have $|E(H)|\geq \frac{1}{2}\left(q^2(q-1)\right)(q^2-2)\geq \frac{1}{2}n^{5/3}+\frac{1}{3}n^{4/3}+C$.

Now it suffices to show H is $K_{3,3}$-free, which is enough to show that for any three distinct vertices (D_i,d_i) with $i\in[3]$, they have at most 2 common neighbors. That is, the system of equations:

$$\begin{cases} N(X+D_1)=xd_1 & (1.1)\\ N(X+D_2)=xd_2 & (1.2)\\ N(X+D_3)=xd_3 & (1.3)\end{cases}$$

has at most 2 solutions $(X,x)\in F_{q^2}\times F_q^*$. Observe that if (X,x) is a solution, then:

1) $X\neq -D_i$, for $i\in[3]$, and

2) $D_i\neq D_j$, for $i\neq j$.

Divide equations (1.1) and (1.2) by equation (1.3), we can get

$$\frac{d_i}{d_3}=\frac{N(X+D_i)}{N(X+D_3)}=N\left(\frac{X+D_i}{X+D_3}\right)=N\left(1+\frac{D_i-D_3}{X+D_3}\right),\ \text{for } i=1,2.$$

Let $Y=\frac{1}{X+D_3}$, $A_i=\frac{1}{D_i-D_3}$, and $b_i=\frac{d_i}{d_3N(D_i-D_3)}$, $i\in[2]$. Then,

$$\begin{cases}(Y+A_1)(Y^q+A_1^q)=N(Y+A_1)=b_1\\ (Y+A_2)(Y^q+A_2^q)=N(Y+A_2)=b_2\end{cases}$$

It is clear that $A_1\neq A_2$ and $A_1^q\neq A_2^q$. Then by lemma 1.6, this system has at most 2 solutions (Y,Y^q). Therefore, we have at most two pairs of (X,x). ∎

Theorem 1.10 (Alon-Rónyai-Szabó)**.** *$H(q,s)$ is $K_{s,(s-1)!+1}$-free. Therefore, for $t \geq (s-1)!+1$,*

$$\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s}).$$

Proof. Exercise (similar to the proof of Theorem 1.7 for $H(q,3)$). ∎

Lecture 2. Random Algebraic Constructions

In this lecture, we use random algebraic/polynomial construction to prove the following result, which gives a weaker bound than Theorem 1.10.

Theorem 2.1. *For any s, there exists $C = C(s)$ such that for any $t \geq C$, $\mathrm{ex}(n, K_{s,t}) = \Omega_{s,t}(n^{2-\frac{1}{s}})$.*

Proof. Let q be a prime power, and F_q be the field of order q. Let $s \geq 4$ be fixed and $q \gg s$. Let $d = s^2 - s + 2$, and $n = q^s$.

Definition 2.2. Let $\vec{X} = \{x_1, x_2, ..., x_s\} \in F_q^s$ and $\vec{Y} = \{y_1, y_2, ..., y_s\} \in F_q^s$. Let $\mathcal{P}$ be all polynomials $f(\vec{X}, \vec{Y})$ of degree at most d in each of $\vec{X}$ and $\vec{Y}$, that is,

$$f(\vec{X}, \vec{Y}) = \sum_{(\vec{a}, \vec{b})} \alpha_{\vec{a}, \vec{b}} \cdot x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \cdot y_1^{b_1} y_2^{b_2} \cdots y_s^{b_s},$$

over all possible choices that $\sum_{i \in [s]} a_i \leq d$ and $\sum_{j \in [s]} b_j \leq d$, where $\alpha_{\vec{a}, \vec{b}} \in F_q$.

We choose a polynomial $f \in \mathcal{P}$ randomly at uniform and use it to define a bipartite graph G_f on partition (F_q^s, F_q^s) with edge set $\{(\vec{X}, \vec{Y}) : f(\vec{X}, \vec{Y}) = 0\}$. Note that $v(G_f) = 2q^s = 2n$. Then by the linearity of expectation, $\mathbb{E}[e(G_f)] = n^2/q = n^{2-1/s}$.

Lemma 2.3. *For any $\vec{u}, \vec{v} \in F_q^s$, $\mathbb{P}[f(\vec{u}, \vec{v}) = 0] = 1/q$.*

Lemma 2.4. *Suppose $r, s \leq \min\{\sqrt{q}, d\}$. Let $U \subseteq F_q^s$ and $V \subseteq F_q^s$ be sets with $|U| = s$ and $|V| = r$. Then*

$$\mathbb{P}[f(\vec{u}, \vec{v}) = 0 \text{ for all } \vec{u} \in U, \text{ and } \vec{v} \in V] = 1/q^{sr}.$$

Fix $U \subseteq F_q^s$ with $|U| = s$. Let $I(\vec{v}) = 1$ if $\vec{v}$ is adjacent to any $\vec{u} \in U$, and otherwise $I(\vec{v}) = 0$. Let $X_U = |N(U)|$. Then $X_U = \sum_{\vec{v}} I(\vec{v})$. We have

$$\begin{aligned}
\mathbb{E}[X_U^d] = \mathbb{E}[(\sum_{\vec{v} \in F_q^s} I(\vec{v}))^d] &= \sum_{\vec{v_1}, \cdots, \vec{v_d} \in F_q^s} \mathbb{E}[I(\vec{v_1}) I(\vec{v_2}) \cdots I(\vec{v_d})] = \sum_{1 \leq r \leq d} \binom{q^s}{r} \frac{1}{q^{rs}} M_r \\
&\leq \sum_{r \leq d} M_r \triangleq M,
\end{aligned}$$

where M_r is defined to the number of surjective mappings from $[d]$ to $[r]$.

Lemma 2.5. *For all s, d, there exists a constant C such that if $f_1(\vec{Y}), f_2(\vec{Y}), ..., f_s(\vec{Y})$ are polynomials over $Y \in F_q^s$ of degree at most d, then*

$$\{\vec{y} \in F_q^s : f_1(\vec{y}) = f_2(\vec{y}) = ... = f_s(\vec{y}) = 0\}$$

has size either at most C or at least $q - C\sqrt{q} \geq q/2$.

By lemma 2.5, if $X_U > C$, then $X_U > q/2$ implies

$$\mathbb{P}[X_U > C] = \mathbb{P}[X_U \geq \frac{q}{2}] = \mathbb{P}[X_U^d \geq (\frac{q}{2})^d] \leq \frac{\mathbb{E}[X_U^d]}{(q/2)^d} \leq \frac{M}{(q/2)^d}.$$

We say a set U of s vertices is *bad* if $X_U > C$. Let u be the number of bad sets U of size s. So we have $\mathbb{E}[u] \leq \binom{q^s}{s} \frac{M}{(q/2)^d} = O(q^{s-2})$ and $\mathbb{E}[e(G_f) - nu] \geq \frac{n^2}{q} - nO(q^{s-2}) \geq \frac{n^2}{2q} = \frac{1}{2} n^{2-1/s}$. Take such a G_f and remove a vertex from every such s-subset to create a new graph G'. We see that G' is $K_{s,C+1}$-free, $v(G') \leq 2n$, and

$$e(G') \geq e(G) - u \cdot n \geq \frac{n^2}{q} - O(q^{s-2})n = (1 - o(1))n^{2-\frac{1}{s}}.$$

∎

Theorem 2.6 (Bukh-Conlon)**.** *For any rational number $r \in (1, 2)$, there is a family of graphs $\mathcal{F}_r$ such that* $\mathrm{ex}(n, \mathcal{F}_r) = \Theta(n^r)$.

Given a rooted tree T with a set R of roots, then p^{th} power $\mathcal{T}^p$ of T is the family of graphs consisting of all possible unions of p distinct labelled copies of T, each of which agree on R.

Definition 2.7. The *density* of a rooted tree (T, R) is defined by

$$\rho_T = \frac{e(T)}{v(T) - |R|}.$$

For any $S \subseteq V(T) \setminus R$, define

$$\rho_S = \frac{\text{The number of edges incident to } S}{|S|}.$$

A rooted tree (T, R) is *balanced* if for any $S \subseteq V(T) \setminus R$, $\rho_S \geq \rho_T$.

Theorem 2.8 (Bukh-Conlon)**.** *For large p,* $\mathrm{ex}(n, \mathcal{T}^p) = \Theta(n^{2-1/\rho_T})$.

Lecture 3. Dependent Random Choice

Theorem 3.1. *Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most r. Then there exists a constant $C = C_H$ such that*

$$ex(n, H) \leq Cn^{2-1/r}$$

Remark 3.2. This theorem was first proved by Füredi (1991) and then was reproved by Alon, Krivelevich and Sudakov (2002).

We will give the proof of Alon-Krivelevich-Sudakov, which has been extended to a powerful probabilistic tool called "dependent random choice". The main idea of this is the following lemma: If G has many many edges, then one can find a large subset A in G such that all small subsets of A have many common neighbors.

Definition 3.3. For $S \subseteq V(G)$, $N(S) = \{w \in V(G) : ws \in E(G) \text{ for every } s \in S\}$.

Lemma 3.4 (Dependent random choice)**.** *Let $u, n, r, m, t \in \mathbb{N}$ and a real number $\alpha \in (0, 1)$ be such that*

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u$$

Then every n-vertex graph G with at least $\frac{\alpha}{2}n^2$ edges contains a subset U of at least u vertices such that every r-element subset S of U has at least m common neighbors.

Proof. Let T be a set of t vertices chosen uniformly at random from $V(G)$ (allowing repetition). Let $A = N(T)$. Then

$$\mathbb{E}[|A|] = \sum_{v \in V} \mathbb{P}[v \in A] = \sum_{v \in V} \mathbb{P}[T \subseteq N(v)] = \sum_{v \in V} \left(\frac{d(v)}{n}\right)^t \geq n \left(\frac{1}{n} \sum_{v \in V} \frac{d(v)}{n}\right)^t \geq n\alpha^t.$$

Call an r-element subset $S \subseteq V(G)$ *bad* if $|N(S)| < m$. Given a bad r-set $S \subseteq V(G)$, we have

$$\mathbb{P}[S \subseteq A] = \mathbb{P}[T \subseteq N(S)] = \left(\frac{|N(S)|}{n}\right)^t < \left(\frac{m}{n}\right)^t.$$

Let s be the number of bad r-subsets in A, so

$$\mathbb{E}[s] < \binom{n}{r}\left(\frac{m}{n}\right)^t,$$

$$\mathbb{E}[|A| - s] \geq n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u.$$

Thus, there exists a choice of T such that $A = N(T)$ satisfies that $|A| - s \geq u$. Let U be obtained from A by deleting one vertex from each bad r-element subset in A. Then we have that $|U| \geq u$ and U satisfies the condition. ∎

Now we can prove the Theorem 3.1.

Proof. (Theorem 3.1) Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most r. We want to show $\text{ex}(n, H) \leq Cn^{2-1/r}$, where $C = C_H$ is a constant. Let G be any n-vertex graph with at least $Cn^{2-1/r}$ edges, where C satisfies

$$n(2Cn^{-1/r})^r - \binom{n}{r}\left(\frac{|A|+|B|}{n}\right)^r \geq |B|.$$

By Lemma 3.4, taking $u = |B|$, $m = |A| + |B|$, $t = r$, $\alpha = 2Cn^{-1/r}$, we see

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u.$$

So there exists a subset U with $|U| \geq u$ such that any r-element subsets of U has at least $m = |A| + |B|$ common neighbors.

We label $A = \{v_1, v_2, ..., v_a\}$ and $B = \{u_1, u_2, ..., u_b\}$. We find any one-to-one mapping $\phi : B \to U$, $u_i \mapsto \phi(u_i)$. Next, we want to extend this ϕ from B to $A \cup B$ and then we can find a copy of H in G. Suppose for $A' = \{v_1, v_2, ..., v_s\}$, we have $\phi : A' \cup B \to V(G)$ such that $H[A' \cup B] \subseteq G[\phi(A') \cup \phi(B')]$. Consider v_{s+1} and $N_H(v_{s+1}) \subseteq B$, we have that $N_H(v_{s+1}) \leq r$. We consider $\phi(N_H(v_{s+1})) \subseteq U$ of size at most r. By the property of U, $\phi(N_H(v_{s+1}))$ has at least $|A|+|B|$ common neighbors in G. Then we can get a vertex $\phi(v_{s+1})$ which is a common neighbor of $\phi(N_H(v_{s+1}))$ but is not in $\phi(A' \cup B)$. Repeatedly, we can extend ϕ to be $\phi : A \cup B \to V(G)$ such that $\phi(A \cup B)$ is a copy of H, a contradiction.

∎

A *subdivision* of a graph H is obtained from H by replacing each edge xy in H with a path $xP_{xy}y$ such that all P_{xy}s are distinct.

Theorem 3.5. *Any n-vertex graph G with at least εn^2 edges has a subdivision of a clique of size at least $\varepsilon^{3/2}n^{1/2}$.*

Proof. This is left to be an exercise. ∎

Lemma 3.6 (Two-sided version of dependent random choice)**.** *Let G be a bipartite graph on $2n$ vertices and with average degree d. Let U, V be two parts of G with $|U| = |V| = n$. If $r, s, t \in \mathbb{N}$ such that*

$$n^{r-s+s^2}d^{-s^2}(t-1)^s < \frac{1}{4}.$$

Then there exist $X \subseteq U$ and $Y \subseteq V$ of size at least $4^{-1/s}n^{1-s}d^s$ satisfying that every r-subset in X (or in Y) has a least t common neighbors in $G(X, Y)$.

A graph H is r-*degenerate* if any one of its subgraphs contains a vertex of degree at most r.

Theorem 3.7. *Let $r \geq 2$ and F be an r-degenerate bipartite graph whose largest part has size t. Then there exists a constant $C = C(F)$ such that*

$$\text{ex}(n, F) \leq C(t-1)^{\frac{1}{2r}}n^{2-\frac{1}{4r}}.$$

Proof. Let C be the constant such that $(\frac{C}{2})^{-4r^2} < \frac{1}{4}$. Let G be a $(2n)$-vertex graph with $e(G) > C(t-1)^{\frac{1}{2r}} n^{2-\frac{1}{4r}}$. Thus, its average degree $d > \frac{C}{2}(t-1)^{\frac{1}{2r}} n^{1-\frac{1}{4r}}$. We know that there exists a subgraph G' of G, which is bipartite with parts U, V of size n and $e(G') \geq e(G)/2$. Let $s = 2r$. It is easy to see that

$$n^{r-s+s^2} d^{-s^2} (t-1)^s < \frac{1}{4},$$

since the choice of C and the inequality implies $4^{-1/s} n^{1-s} d^s \geq t$. By Lemma 3.6, we obtain that there exist $X \subseteq U$ and $Y \subseteq V$ of size at least $4^{-1/s} n^{1-s} d^s$ satisfying that every r-subset in X(or in Y) has a least t common neighbors in $G(X, Y)$.

Let F be a bipartite graph on partition $A \cup B$. Our goal is to construct an embedding $f : V(F) \to V(G)$ by placing images of vertices from A into X, and images of vertices of B into Y. To construct the desired embedding, we proceed according to the chosen order $(v_1, \ldots, v_h)$ of the vertices of F. If the current vertex $v_i \in V(F), i \in [h]$ is a vertex from A, we first locate the images $f(v_j)$, $j < i$, of the already embedded neighbours of v_i in B. The set $\{f(v_j) : j < i, (v_j, v_i) \in E(H)\}$ is a subset of Y of cardinality at most r. It therefore has at least t common neighbours in X, and obviously not all of them have already been used in the embedding. We pick one unused vertex w and set $f(v_i) = w$. If $v_i \in B$, we can repeat the above argument, interchanging the roles of X and Y. We can find a copy of F in (X, Y), a contradiction. So, we have

$$\mathrm{ex}(n, F) \leq C(t-1)^{\frac{1}{2r}} n^{2-\frac{1}{4r}}.$$

∎

Corollary 3.8. *For any bipartite graph F, let $d_F = \max_{F' \subseteq F} \frac{2e(F')}{v(F')}$. Then*

$$\mathrm{ex}(n, F) = O(n^{2-\frac{1}{4\lfloor d_F \rfloor}}) = O(n^{2-\frac{1}{4d_F}}).$$

Hint: It holds since F is $\lfloor d_F \rfloor$-degenerate.

Corollary 3.9. *For bipartite graph F, let*

$$c_F = \min_{F' \subseteq F} \frac{v(F')}{e(F')}$$

and

$$c_F^* = \min_{F' \subseteq F, e(F') \geq 2, \delta(F') \geq 1} \frac{v(F') - 2}{e(F') - 1}.$$

Then

$$\mathrm{ex}(n, F) = \Omega(n^{2-c_F^*}) \geq \Omega(n^{2-c_F}).$$

Lecture 4. Lower bounds on Ramsey numbers

4.1 Lovász Local Lemma

Theorem 4.1 (Lovász Local Lemma (Symmetric Version))**.** *Let $\{A_i\}_{i\in[k]}$ be a family of random events. For any i, $\mathbb{P}[A_i] \le p$. Any event is independent of all other events except for d of them. If $ep(d-1) < 1$, then the probability that all events' complements occur simultaneously is greater than 0, that is:*

$$\mathbb{P}\left[\bigcap A_i^c\right] > 0.$$

We focus on the asymmetric version of the Lovász Local Lemma, which is stronger than the symmetric version. Firstly, we define the following auxiliary graph.

Definition 4.2. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a family of random events in the probability space Ω. Let $D := D_{\mathcal{A}}$ be the *dependence graph* with vertex set $V(D) = \mathcal{A}$ and edge set $E(D) = \{A_iA_j :$ A_i and A_j are dependent for each $i, j \in [k]\}$.

Theorem 4.3 (Lovász Local Lemma (Asymmetric Version))**.** *Given a probability space $(\Omega, \mathbb{P})$, the event collection $\mathcal{A}$ and the dependence graph D in Definition 4.2. Denote the neighborhood set of A_i in D by $N(A_i)$. If there exists a mapping $f : \mathcal{A} \to [0, 1)$ satisfying that*

$$\mathbb{P}[A_i] \le f(A_i) \prod_{B\in N(A_i)} (1 - f(B))$$

holds for each $A_i \in \mathcal{A}$, then the following holds:

$$\mathbb{P}\left[\bigcap A_i^c\right] > 0.$$

4.2 Applications of Lovász Local Lemma

The Ramsey number $r(k, \ell)$ is the smallest integer N such that any red-blue edge-coloring of K_N contains a red K_k or a blue K_ℓ.

Remark:

- Ramsey's Theorem [3]: The Ramsey number exists.
- Erdős and Szekeres [2]: $r(k, \ell) \le \binom{k+\ell-2}{k-1}$. If $k = \ell$, then this yields $r(k, k) \le 4^k$.
- Campos, Griffiths, Morris and Sahasrabudhe [1]: There exists $\varepsilon > 0$ such that $r(k, k) \le (4 - \varepsilon)^k$.

Theorem 4.4.

$$r(3, \ell) = \Omega\left(\frac{\ell^2}{\log^2 \ell}\right).$$

Proof. Consider a random edge-coloring of the complete graph K_n, where each edge is colored red with probability p and blue with probability $1 - p$ independently. Our goal is to obtain that with positive probability there is a coloring without a red triangle and without a blue K_ℓ, since this would establish the lower bound $r(3, \ell) > n$.

For each 3-element set $T \subseteq [n]$, let A_T be the event that T induces a red K_3. Note that for each T, we have $\mathbb{P}[A_T] = p^3$. For each ℓ-element set $S \subseteq [n]$, let B_S be the event that S induces a blue K_ℓ. Note that for each ℓ, we have $\mathbb{P}[B_S] = (1-p)^{\binom{\ell}{2}}$. Let us now define a dependence graph D for these events. We join two events of the form A_T or B_S, if the corresponding sets S or T share an edge. Now we can bound the degrees in this graph,

$$\begin{cases} \deg(A_T) \le \binom{3}{2}(n-3) + \binom{3}{2}\binom{n-3}{\ell-2} \le 3n + \binom{n}{\ell} \text{ for each } A_T, \\ \deg(B_S) \le \binom{\ell}{2}n + \binom{\ell}{2}\binom{n-3}{\ell-2} \le \binom{\ell}{2}n + \binom{n}{\ell} \text{ for each } B_S. \end{cases}$$

Define a mapping $f : \{A_T : T \in \binom{[n]}{3}\} \cup \{B_S : S \in \binom{[n]}{\ell}\} \to [0,1)$ such that $x := f(A_T)$ and $y := f(B_S)$. In order to use Lovász Local Lemma, we need to find positive real numbers $x, y \in (0,1)$ such that

$$\begin{cases} p^3 \le x(1-x)^{3n}(1-y)^{\binom{n}{\ell}}, \\ (1-p)^{\binom{\ell}{2}} \le y(1-x)^{n\binom{\ell}{2}}(1-y)^{\binom{n}{\ell}}, \end{cases}$$

Let us now try to find such $p, x, y \in (0,1)$ for sufficiently large n. We choose $y = \frac{1}{\binom{n}{\ell}}$, then $(1-y)^{\binom{n}{\ell}} \approx \frac{1}{e}$. Furthermore, we observe that p and x need to fulfill the following inequalities:

$$p^3 \le x(1-x)^{3n}(1-y)^{n/2} \le x,$$

$$e^{-p\binom{\ell}{2}} \approx (1-p)^{\binom{\ell}{2}} \le y(1-x)^{\binom{\ell}{2}n}(1-y)^{n/2} \le (1-x)^{\binom{\ell}{2}n} \approx e^{-xn\binom{\ell}{2}}.$$

Hence we need $p \ge xn \ge p^3 n$. Therefore $p \le \frac{1}{\sqrt{n}}$ and $x \ge p^3$. Finally, for the second condition, we note

$$e^{-p\binom{\ell}{2}} \approx (1-p)^{\binom{\ell}{2}} \le y(1-x)^{\binom{\ell}{2}n}(1-y)^{n/2} \le y = \frac{1}{\binom{n}{\ell}} \approx e^{-\ell \log n},$$

hence $p\ell^2 \ge p\binom{\ell}{2} \ge \ell \log n$ and therefore $\ell \ge \frac{1}{p}\log n \ge \sqrt{n}\log n$.

Motivated by this we may assume $\ell \ge 20\sqrt{n}\log n$ and choose $y = \frac{1}{\binom{n}{\ell}}, x = \frac{1}{9n^{3/2}}$ and $p = \frac{1}{3\sqrt{n}}$. After choosing the constants, we can give the full proof. For sufficiently large n, we have

$$(1-y)^{\binom{n}{\ell}} = \left(1 - \frac{1}{\binom{n}{\ell}}\right)^{\binom{n}{\ell}} \ge e^{-1.01},$$

$$(1-x)^{3n} = \left(1 - \frac{1}{9n^{3/2}}\right)^{3n} \ge 1 - \frac{1}{3\sqrt{n}} \ge e^{-0.01}.$$

Thus,

$$p^3 = \frac{1}{27n^{3/2}} \le \frac{1}{9n^{3/2}} \cdot \frac{1}{3} \le \frac{1}{9n^{3/2}}e^{-1.02} \le x(1-x)^{3n}(1-y)^{\binom{n}{\ell}},$$

which establishes the first desired inequality.

For the second inequality, for sufficiently large n, we get

$$(1-x)^{\binom{l}{2}n} \ge e^{-2xn\binom{l}{2}} \ge e^{-\frac{2}{9\sqrt{n}}\binom{\ell}{2}}.$$

Furthermore, using $\ell \ge 20\sqrt{n}\log n$, we have

$$y = \frac{1}{\binom{n}{\ell}} \ge \frac{1}{n^\ell} = e^{-l\log n} \ge e^{-l^2\frac{1}{20\sqrt{n}}} \ge e^{-\ell(\ell-1)\frac{1}{19\sqrt{n}}} \ge e^{-\frac{1}{9\sqrt{n}}\binom{\ell}{2}+1.01}.$$

Hence

$$(1-p)^{\binom{\ell}{2}} \le e^{-p\binom{\ell}{2}} = e^{-\frac{1}{3\sqrt{n}}\binom{\ell}{2}} = e^{-\frac{1}{9\sqrt{n}}\binom{\ell}{2}+1.01} e^{-\frac{2}{9\sqrt{n}}\binom{\ell}{2}} e^{-1.01} \le y(1-x)^{\binom{\ell}{2}n}(1-y)^{\binom{n}{\ell}},$$

which verifies the second desired inequality.

By Lemma 4.3, we obtain that

$$\mathbb{P}\left[\bigcap_{T\in\binom{[n]}{3}} A_T^c \cap \bigcap_{S\in\binom{[n]}{\ell}} B_S^c\right] > 0.$$

So for $\ell \ge 20\sqrt{n}\log n$, we can find p, x, y such that there exists a 2 edge-coloring of K_n such that there is no red triangle and there is no blue K_ℓ. This implies $r(3,\ell) > n$.

Note that $n \le \frac{\ell^2}{(40\log\ell)^2}$ implies

$$20\sqrt{n}\log n \le 20\frac{\ell}{40\log\ell}\log\ell^2 = \ell.$$

Therefore we have $r(3,\ell) \ge \frac{\ell^2}{(40\log\ell)^2}$. ∎

Theorem 4.5 (Erdős' Lower Bound)**.** *Let $C \ge 1$ and $p_C \in (0,1/2]$ be the unique solution to $C = \frac{\log p_C}{\log(1-p_C)}$. Let $M_C = p_C^{-1/2}$. Then $r(\ell, C\ell) = \Omega(\ell\cdot M_C^\ell)$. In particular, when $C=1$, we have $p_C = 1$ and $r(\ell,\ell) = \Omega(\ell\sqrt{2}^\ell)$.*

Proof. Let $p\in(0,1/2]$. Consider a random edge-coloring of the complete graph K_n, where each edge is independently colored red with probability p and blue with probability $1-p$. Let

$$f(n,p) := A(n,p) + B(n,p) \text{ where } A(n,p) = \binom{n}{\ell}p^{\binom{\ell}{2}} \text{ and } B(n,p) = \binom{n}{C\ell}(1-p)^{\binom{C\ell}{2}}.$$

Note that

$$\mathbb{P}[\text{There exists a red } K_\ell \text{ or a blue } K_{C\ell}] \le f(n,p).$$

Hence, if $f(n,p) = 1 - o_\ell(1)$, then there exists at least one such coloring with no red K_ℓ and no blue $K_{C\ell}$, implying $r(\ell, C\ell) > n$. It thus suffices to find the maximum value of $n = n(p)$ such that $f(n,p) = o_\ell(1)$. Assume this maximum is achieved at $p_0 = p_{C,\ell}$. Then,

$$\frac{\partial f(n,p_0)}{\partial p} = \frac{\binom{\ell}{2}}{p_0}\binom{n}{\ell}p_0^{\binom{\ell}{2}} - \frac{\binom{C\ell}{2}}{1-p_0}\binom{n}{C\ell}(1-p_0)^{\binom{C\ell}{2}} = 0.$$

Thus, we have $\log A(n,p_0) = \log B(n,p_0) + O(\log\ell)$. Solving this along with $A(n,p_0)+B(n,p_0) = 1-o_\ell(1)$, we obtain that $\log A(n,p_0) = O(\log\ell)$ and $\log B(n,p_0) = O(\log\ell)$. Therefore, we obtain that

$$\begin{cases} -\log p_0 = \frac{2\log(en/\ell)}{\ell-1} + O\left(\frac{\log\ell}{\ell^2}\right), \\ -\log(1-p_0) = \frac{2\log(en/C\ell)}{C\ell-1} + O\left(\frac{\log\ell}{\ell^2}\right). \end{cases}$$

We then derive that $p_0 = p_C + O(1/\ell)$, where the constant p_C satisfies $C = \frac{\log p_C}{\log(1-p_C)}$. It follows directly from the above that $n = \frac{\ell}{e}\cdot p_0^{-(\ell-1)/2}\cdot e^{O\left(\frac{\log\ell}{\ell}\right)} = \Theta(\ell)\cdot(p_C + O(1/\ell))^{-\ell/2} = \Theta(\ell\cdot M_C^\ell)$, where $M_C := p_C^{-1/2}$. This establishes $r(\ell, C\ell) = \Omega(\ell\cdot M_C^\ell)$. ∎

References

[1] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe. An exponential improvement for diagonal Ramsey. *arXiv:2303.09521*, 2025.

[2] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compos. Math.*, 2:463–470, 1935.

[3] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930.

Extensive reading

The following two papers provide relevant extensive reading materials:

1. Noga Alon, Lajos Rónyai and Tibor Szabó, Norm-graphs: variations and applications, Journal of Combinatorial Theory, Series B 76 (1999), 280–290.

2. B. Bukh and D. Conlon, Rational exponents in extremal graph theory, Journal of the European Mathematical Society, 20 (2018), 1747–1757.