2025 BUPT Summer School - Course III
Regularity Methods and its Applications

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Lecture 1 The regularity method and the blowup lemma

As we all know, the regularity methods are some of the most powerful tools in combinatorics, which played a central role in graph theory, functional Analysis, ergodic theory and so on. Here, we firstly get to know one of the most classical applications of regularity lemma.

Lemma 1.1 (Triangle removal lemma)**.** *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that the following holds for large n. If G is an n-vertex graph with at most δn^3 triangles, then G can be made K_3-free by removing at most εn^2 edges.*

Next we can obtain the general result by extending triangle to any graph H.

Lemma 1.2 (Graph removal lemma)**.** *For any graph H and any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that any graph on n vertices which contains at most $\delta n^{v(H)}$ copies of H may be made H-free by removing at most εn^2 edges.*

Let $G = (V, E)$ be a graph. For disjoint sets $X, Y \subseteq V(G)$, the *edge-density* between X and Y is

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

Definition 1.3 (ε-regular)**.** *Given a graph G and some $\varepsilon > 0$, $V_1, V_2 \subseteq V(G)$, $V_1 \cap V_2 = \varnothing$. A pair (V_1, V_2) is called ε-regular if for any $A \subseteq V_1$ and $B \subseteq V_2$ with $|A| \geq \varepsilon |V_1|$, $|B| \geq \varepsilon |V_2|$, then $|d(A,B) - d(V_1,V_2)| < \varepsilon$.*

In particular, for convenience, we call a pair (V_1, V_2) (ε, d)-regular if it is ε-regular and $d(V_1, V_2) = d$.

Exercise 1.4. *Given $\varepsilon, c > 0$ and $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ with $|V_i'| \geq c|V_i|$. If (V_1, V_2) is ε-regular, then (V_1', V_2') is "also regular"* (that is, $\max\{2\varepsilon, \varepsilon/c\}$-regular).

Lemma 1.5 (K_3-Counting lemma)**.** *For every $\varepsilon > 0$, the following holds for large n. Suppose that there exist three disjoint vertex sets V_1, V_2, V_3 with $|V_i| \geq n$ such that for any $i, j \in [3]$, (V_i, V_j) is ε-regular and $d(V_i, V_j) \geq 2\varepsilon$. Then $G[V_1, V_2, V_3]$ contains at least $(1 - 2\varepsilon)(d_{12} - \varepsilon)(d_{13} - \varepsilon)(d_{23} - \varepsilon)|V_1||V_2||V_3|$ triangles.*

Proof. Take $V_1' \subseteq V_1$ such that $v \in V_1'$ if and only if $d(v, V_2) < (d_{12} - \varepsilon)|V_2|$ or $d(v, V_3) < (d_{13} - \varepsilon)|V_3|$. Then we claim that $|V_1'| \leq 2\varepsilon|V_1|$. Otherwise, if there exists a vertex set $V_{12} \subseteq V_1$ with $\varepsilon|V_1|$ vertices such that for any $v \in V_{12}$, $d(v, V_2) < (d_{12} - \varepsilon)|V_2|$, then we have $d(V_{12}, V_2) < \frac{(d_{12}-\varepsilon)|V_2||V_{12}|}{|V_{12}||V_2|} = d_{12} - \varepsilon$. But by the definition of ε-regular, we have $d(V_{12}, V_2) > d_{12} - \varepsilon$, a contradiction. Similarly, if there exists a vertex set $V_{13} \subseteq V_1$ with $\varepsilon|V_1|$ vertices such that for any $v \in V_{13}$, $d(v, V_3) < (d_{13} - \varepsilon)|V_3|$, then we have $d(V_{13}, V_3) < \frac{(d_{13}-\varepsilon)|V_3||V_{13}|}{|V_{13}||V_3|} = d_{13} - \varepsilon$. But by the definition of ε-regular, we have $d(V_{13}, V_3) > d_{13} - \varepsilon$, a contradiction. Thus, we derive that $|V_1'| \leq 2\varepsilon|V_1|$.

Now we consider the pair $(N(u) \cap V_2, N(u) \cap V_3)$. Note that $d(N(u) \cap V_2, N(u) \cap V_3) \in (d_{23} - \varepsilon, d_{23} + \varepsilon)$ because $N(u) \cap V_2 \subseteq V_2$ and $N(u) \cap V_3 \subseteq V_3$. Take any vertex $u \in V_1 \smallsetminus V_1'$. Since $|N(u) \cap V_2| = d(u, V_2) \geq (d_{12} - \varepsilon)|V_2| \geq \varepsilon|V_2|$ and $|N(u) \cap V_3| = d(u, V_3) \geq (d_{13} - \varepsilon)|V_3| \geq \varepsilon|V_3|$, we have

$$\begin{aligned} e(N(u) \cap V_2, N(u) \cap V_3) &\geq (d_{23} - \varepsilon) \cdot |N(u) \cap V_2| \cdot |N(u) \cap V_3| \\ &\geq (d_{23} - \varepsilon)(d_{12} - \varepsilon)|V_2|(d_{13} - \varepsilon)|V_3|. \end{aligned}$$

Sum over all $u \in V_1 \smallsetminus V_1'$, we get the number of K_3 in G is at least

$$(1 - 2\varepsilon)|V_1| \cdot (d_{23} - \varepsilon)(d_{12} - \varepsilon)|V_2|(d_{13} - \varepsilon)|V_3| = (1 - 2\varepsilon)(d_{12} - \varepsilon)(d_{13} - \varepsilon)(d_{23} - \varepsilon)|V_1||V_2||V_3|. \qquad \square$$

Remark 1.6.

- *Extend to K_r-counting in regular r-tuples by induction.*
- *Extend to F-counting in regular $\mathcal{X}(F)$-tuples, where $\mathcal{X}(F)$ is the chromatic number of G.*

Theorem 1.7 (Regularity lemma)**.** *For every $\varepsilon > 0$, $t \in \mathbb{N}$, there exist $N = N(\varepsilon, t)$ and $T = T(\varepsilon, t)$ such that the following holds for every $n \geq N$. Every n-vertex graph G admits an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_r$ with $t \leq r \leq T$,*

(1) $|V_i| = |V_j|$ for $1 \leq i, j \leq r$,

(2) $|V_0| \leq \varepsilon n$,

(3) (V_i, V_j) is ε-regular for all but at most εr^2 pairs with $i, j \in [r]$.

Remark 1.8.

- *Only meaningful for dense graphs.*
- *$T = T(\varepsilon, t)$ is the upper bound of r, guaranteeing the "quality" of partition, but T is very large, which is $2^{2^{2^{\cdot^{\cdot^{2}}}}}$, where the height of the tower is a function of ε. Notice that the number of index is a function of ε and Gowers showed that this is unavoidable.*

- *Sometimes (or most of the time), you want to choose t large.*

Proof of triangle removal lemma. For every $\varepsilon > 0$, let ε be small and n be large. Suppose that G is a graph with less than δn^3 triangles.

Apply the regularity lemma with $t = 4/\varepsilon$ and $\delta = \frac{\varepsilon^3}{128T^3}$. Let $V_0 \cup V_1 \cup \cdots \cup V_r$ be the $\varepsilon/4$-regular partition with $t \le r \le T$.

Next, we will perform the following operation:

- removing all edges incident to V_0;
- removing all edges between irregular pairs;
- removing all edges inside each V_i with $i \in [r]$;
- removing all edges for (V_i, V_j) with $d(V_i, V_j) < \varepsilon/2$.

Thus, we removed at most

$$\begin{aligned}
&\frac{\varepsilon n}{4} \cdot (n-1) + \frac{\varepsilon r^2}{4} \cdot \left(\frac{n - |V_0|}{r}\right)^2 + r \cdot \binom{(n-|V_0|)/r}{2} + \binom{r}{2} \cdot \frac{\varepsilon}{2} \left(\frac{n-|V_0|}{r}\right)^2 \\
\le\ & \frac{\varepsilon n}{4} \cdot n + \frac{\varepsilon r^2}{4} \cdot \left(\frac{n}{r}\right)^2 + r \cdot \left(\frac{n}{r}\right)^2 + \binom{r}{2} \cdot \frac{\varepsilon}{2} \left(\frac{n}{r}\right)^2 \\
=\ & \frac{\varepsilon n^2}{4} + \frac{\varepsilon n^2}{4} + \frac{n^2}{2r} + \frac{\varepsilon n^2}{4} \\
=\ & \frac{3\varepsilon n^2}{4} + \frac{n^2}{2r} \le \varepsilon n^2
\end{aligned}$$

edges since $r \ge t = 4/\varepsilon$.

Let G' be the resulting graph. Now note that if $G' \supseteq K_3$, then there exist i, j, k such that this K_3 belongs to V_i, V_j, V_k and $(V_i, V_j), (V_i, V_k), (V_j, V_k)$ are all $\varepsilon/4$-regular with density $\ge \varepsilon/2$. Then the K_3-Counting lemma implies that $G'[V_i, V_j, V_k]$ has at least $(1-\varepsilon/2)(d_{ij} - \varepsilon/4)(d_{jk} - \varepsilon/4)(d_{ik} - \varepsilon/4)|V_i||V_j||V_k| \ge (1-\varepsilon/2) \cdot (\varepsilon/4)^3 \cdot \left(\frac{n - \varepsilon n/4}{r}\right)^3 > \frac{\varepsilon^3}{128T^3} n^3 = \delta n^3$ triangles, which contradicts with assumption. Thus, G' is K_3-free, that is, we obtain a K_3-free graph G' by removing at most εn^2 edges from G. □

Remark 1.9. *Can we get better dependency between ε and δ? Improved bounds obtained by Fox (2011), by iterating Frieze-Kannan weak regularity.*

Other notable applications:

- $RT(K_4)$.
- If $\Delta(H) \le \Delta$, then $r(H) = O(|H|)$.
- Alon-Yuster theorem (by applying Blow-up lemma).

Application: Ramsey-Turán Theory

Question: If graph G is K_4-free and $\alpha(G) = o(n)$, then how many edges can G have? Szemerédi presented the following result.

Theorem 1.10 (Szemerédi)**.** *For any $\varepsilon > 0$, there exists $\alpha > 0$ such that the following holds for large n. If G is a K_4-free n-vertex graph and $\alpha(G) \leq \alpha n$, then $e(G) \leq (\frac{1}{8} + \varepsilon)n^2$.*

Proof. Let $\alpha = \frac{2\varepsilon^2}{25T}, t = \frac{5}{\varepsilon}$ and regularize graph G with $\varepsilon/5$. Then we get the following partition:

- $|V_0| \leq \varepsilon n/5$,
- For all $1 \leq i < j \leq r$, $|V_i| = |V_j|$,
- (V_i, V_j) is $\varepsilon/5$-regular for all but at most $\varepsilon r^2/5$ pairs with $i, j \in [r]$.

Claim 1.11. *If (V_i, V_j) is ε-regular, then $d(V_i, V_j) < \frac{1}{2} + \frac{2\varepsilon}{5}$.*

Proof. Suppose that $d(V_i, V_j) \geq \frac{1}{2} + \frac{2\varepsilon}{5}$. Let $V_i' \subseteq V_i$ be the vertices that have degree $< (\frac{1}{2} + \frac{\varepsilon}{5})|V_j|$ to V_j. Then $|V_i'| \leq \frac{\varepsilon}{5}|V_i|$. Thus, we have $|V_i \setminus V_i'| \geq (1 - \frac{\varepsilon}{5})|V_i| \geq (1 - \frac{\varepsilon}{5}) \cdot (1 - \frac{\varepsilon}{5})\frac{n}{r} \geq \frac{n}{2T} > \alpha n$. Since $\alpha(G) \leq \alpha n$, we can pick an edge uv in $V_i \setminus V_i'$. Since $d(u, V_j), d(v, V_j) \geq (\frac{1}{2} + \frac{\varepsilon}{5})|V_j|$, we get

$$|N(u) \cap N(v) \cap V_j| \geq \frac{2\varepsilon}{5}|V_j| > \frac{2\varepsilon}{5} \cdot (1 - \frac{\varepsilon}{5})\frac{n}{r} > \frac{\varepsilon n}{5T} > \alpha n.$$

Then we can pick an edge in $N(u) \cap N(v)$, giving a $K_4 \subseteq G$, a contradiction. □

Next we define a d-Reduced graph R: Let R be a graph on $[r]$ such that $ij \in E(R)$ if and only if (V_i, V_j) is (ε, d')-regular with $d' \geq d$.

Let $d = 3\varepsilon/5$ and R be the d-reduced graph of the partition $(V_1, \ldots, V_r)$.

Claim 1.12. *R is K_3-free.*

Proof. Suppose not. Without loss of generality, there are three vertices 1,2,3 from V_1, V_2, V_3 forming a $K_3 \subseteq R$. Let $V_1' \subseteq V_1$ be vertex set such that for any vertex $v \in V_1'$, $d(v, V_2) < (d - \frac{\varepsilon}{5})|V_2|$ or $d(v, V_3) < (d - \frac{\varepsilon}{5})|V_3|$. Then $|V_1'| \leq \frac{2\varepsilon}{5}|V_1|$.

Now we take a vertex $u \in V_1 \setminus V_1'$ and let $X = N(u) \cap V_2$, $Y = N(u) \cap V_3$. Note that $|X| \geq d(u, V_2) \geq (d - \frac{\varepsilon}{5})|V_2| \geq \frac{2\varepsilon}{5}|V_2|$. Let $X' \subseteq X$ be the vertex set such that for any vertex $w \in X'$, $d(w, Y) < (d - \varepsilon)|Y|$. By regularity, we get $|X'| \leq \varepsilon|V_2|$, which implies that $|X \setminus X'| \geq |X| - \varepsilon|V_2| \geq \varepsilon|V_2|$.

Next we take any $v_2 \in X \setminus X'$, then

$$d(v_2, Y) \geq \left(d - \frac{\varepsilon}{5}\right)|Y| \geq \left(d - \frac{\varepsilon}{5}\right) \cdot \left(d - \frac{\varepsilon}{5}\right)|V_3| \geq \left(d - \frac{\varepsilon}{5}\right)^2 \left(\frac{n - \frac{\varepsilon n}{5}}{r}\right) \geq \frac{2\varepsilon^2}{25T}n > \alpha n.$$

Then we can pick an edge in $N(v_2, Y) = N(vv_2, V_3)$, giving a $K_4 \subseteq G$, a contradiction. □

Now we compute $e(G)$ by counting the following five parts:

- all edges incident to V_0, which is at most $\varepsilon n^2/5$,
- all edges between irregular pairs, which is at most $\frac{\varepsilon r^2}{5} \cdot (\frac{n}{r})^2 = \varepsilon n^2/5$,
- all edges inside each V_i with $i \in [r]$, which is at most $r \cdot \binom{n/r}{2} \le \frac{n^2}{2r} \le \frac{n^2}{2t}$,
- all edges for (V_i, V_j) with $d(V_i, V_j) < d$, which is at most $\binom{r}{2} \cdot d(\frac{n}{r})^2 \le \frac{d}{2} n^2$,
- all edges in R: Since R is K_3-free, by Mantel's theorem, $e(R) \le \frac{r^2}{4}$ and each edge has density less than $\frac{1}{2} + \frac{2\varepsilon}{5}$. So the number of edges in R is at most $\frac{r^2}{4} \cdot (\frac{1}{2} + \frac{2\varepsilon}{5})(\frac{n}{r})^2 = (\frac{1}{8} + \frac{\varepsilon}{10}) n^2$.

Adding all these up, we have

$$e(G) \le 2\varepsilon n^2/5 + \frac{n^2}{2t} + \frac{dn^2}{2} + \left(\frac{1}{8} + \frac{\varepsilon}{10}\right) n^2 \le \left(\frac{1}{8} + \varepsilon\right) n^2.$$

□

Remark 1.13.

- *Bollobás-Erdős found a graph saying that the bound* $1/8$ *is sharp.*
- *This theorem appeared before the Regularity lemma.*

Exercise 1.14. *Prove Erdős–Stone–Simonovits Theorem: Fix graph* H *with at least one edge. Then*

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

Lecture 2 The regularity method and the blowup lemma

In Lecture 1 we use an embedding scheme for proving the K_3-counting lemma. If we do the embedding a bit more carefully, then we can embed a subgraph of small linear size.

Lemma 2.1 (Graph embedding lemma). *For any $\Delta, n, m \in \mathbb{N}$, let $\varepsilon < \frac{(d-\varepsilon)^\Delta}{\Delta+2}$ and $m \le \varepsilon n$. Suppose that G is a graph satisfying $v(G) = V_1 \cup \cdots \cup V_r$ with $|V_i| \ge n$ for $i \in [r]$ and (V_i, V_j) is (ε, d')-regular, where $d' \ge d$ and $i \ne j \in [r]$. Let H be an h-vertex r-partite graph with partition $X_1 \cup \cdots \cup X_r$ with maximum degree Δ and $|X_i| \le m$ for $i \in [r]$. Then $H \subseteq G$.*

Proof. Let $v(H) = \{x_1, \dots, x_h\}$, and let $\phi(i) \in [r]$ such that $x_i \in X_{\phi(i)}$. We will construct an embedding f by defining $V_1 = f(x_1), V_2 = f(x_2), \dots, v_h = f(x_h)$. Denote $C_i(j)$ to be the set of possible candidates of v_j after we determine $v_1, v_2, \dots, v_{i-1}$.

Now we embed X_i to V_i. Initially, $C_i(j) = V_{\phi(j)}$. Suppose that we have determined $v_1, v_2, \dots, v_{i-1}$ and we have $|C_i(j)| \ge (\Delta+1)\varepsilon n$ for all $j \ge i$. Then we need to select v_i from $C_i(i)$. Consider $A = \{x_j \in N_H(x_i) : j > i\} = \{x_{s_1}, \dots, x_{s_p}\}$, $p \le \Delta$. Since $(V_{\phi(i)}, V_{\phi(s_\ell)})$ is (ε, d')-regular for $\ell \in [p]$, all but at most εn vertices in $C_i(i)$ have at least $(d-\varepsilon)|C_i(s_\ell)|$ neighbors in $C_i(s_\ell)$ for $\ell \in [p]$. Thus, there exist at least $|C_i(i)| - \Delta\varepsilon n$ vertices in $C_i(i)$ have at least $(d-\varepsilon)|C_i(S_\ell)|$ neighbors in $C_i(s_\ell)$ for $\ell \in [p]$. Among them, at most $m-1 \le \varepsilon n - 1$ vertices are in $\{v_1, v_2, \dots, v_{i-1}\}$. Then we can choose one vertex as $v_i \notin \{v_1, v_2, \dots, v_{i-1}\}$ that has at least $(d-\varepsilon)|C_i(S_\ell)|$ neighbors in $C_i(s_\ell)$ for $\ell \in [p]$.

Let $v_i = f(x_i)$. Next, we will make the following update. Let

$$C_{i+1}(j) = \begin{cases} C_i(j) \cap N_G(v_i) & \text{if } x_j \sim x_i, \\ \\ C_i(j) & \text{if } x_j \nsim x_i. \end{cases}$$

Since x_j has at most Δ neighbors, $|C_{i+1}(j)| \ge n(d-\varepsilon)^\Delta \ge (\Delta+1)\varepsilon n$ throughout the process. Thus, we can always choose v_i for $i \le h$. Then we obtain an embedding of $H \subseteq G$ satisfying the partition. □

Application: Ramsey Theory

Given a graph H, let $r(H) = \min n$ such that any edge-coloring of K_n contains a monochromatic copy of H.

Theorem 2.2 (Chvátal - Rödl - Szemerédi - Trotter). *Fix Δ and let H be a graph with $\Delta(H) \le \Delta$. Then there exists $c = c(\Delta)$ such that $r(H) \le c|H|$.*

Proof. Let $k = r(K_{\Delta+1})$. Take $\varepsilon = \frac{1}{2^{\Delta+1}k}$, $t = \Delta+1$. Assume that $N = N(\varepsilon, t)$, $T = T(\varepsilon, t)$ as defined in regularity lemma. Let $c = c(\Delta) = \max\{3T/\varepsilon, N\}$. Take $n > c|H| = 3T|H|/\varepsilon$. Next, we need to show that every 2-edge-coloring of $E(K_n)$ contains a monochromatic copy of H.

Let G be the red graph. Applying the regularity lemma to G with ε, t, we can obtain an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_r$ for G with $|V_i| \ge (1-\varepsilon)\frac{n}{r} \ge \frac{2|H|}{\varepsilon}$ for $i \in [r], t \le r \le T$. Consider a graph R on $[r]$ such that $ij \in E(R)$ if and only if (V_i, V_j) is ε-regular. Then

$$|E(R)| \ge \binom{r}{2} - \varepsilon r^2 \ge (1-3\varepsilon)\binom{r}{2} > (1 - \frac{1}{k-1})\binom{r}{2}.$$

By Turán's theorem, R contains a copy of K_k.

Now color the edges of K_k in the following way: color ij red if $d(V_i, V_j)$ has red density $\ge \frac{1}{2}$, and color blue otherwise. (Note that all pairs are ε-regular.) By the definition of $k = r(K_{\Delta+1})$, there exists a monochromatic copy of $K_{\Delta+1}$ in this K_k. Then, $V_1, V_2, \ldots, V_{\Delta+1}$ are red(or blue) regular $(\Delta+1)$-tuple and the pair (V_i, V_j) is ε-regular for $i, j \in [\Delta+1]$. Now we define graph G' as the red(or blue) graph on $V_1, \ldots, V_{\Delta+1}$. Then G' is a $(\Delta+1)$-partite graph on $V_1, \ldots, V_{\Delta+1}$ such that (V_i, V_j) is (ε, d)-regular with $d \ge \frac{1}{2}$ and $|H| \le \varepsilon |V_i|/2$. Apply the Graph embedding lemma to G' with $d = \frac{1}{2}$ and $m = |H|$, we can find a copy of H in G', which gives a monochromatic copy of H. Thus, there exists $c = c(\Delta)$ such that $r(H) \le c|H|$. □

Question: What about embedding large subgraphs or spanning subgraphs?

Definition 2.3. *Let G be a graph. A disjoint pair (A, B) of vertices is (ε, d)-super-regular if it's ε-regular, $d(A, B) \ge d$ and $d(a, B) \ge (d-\varepsilon)|B|, d(b, A) \ge (d-\varepsilon)|A|$ for all $a \in A, b \in B$.*

Lemma 2.4. *Let $2\varepsilon \le d \le 1$ and $n \ge 2/\varepsilon$. Let G be a graph. If (A, B) is (ε, d)-super-regular in G with $|A| = |B| = n$, then $G[A, B]$ contains a perfect matching.*

Theorem 2.5 (Blow-up lemma (Komlós-Sárközy-Szemerédi))**.** *Let $0 < \frac{1}{n} \ll \varepsilon \ll \frac{1}{r}, d_0, \frac{1}{\Delta} \le 1$. Suppose that H is an n-vertex graph satisfying vertex partition $X_1 \cup \cdots \cup X_r$ with $\Delta(H) \le \Delta$. Let G be a graph with partition $V_1 \cup \cdots \cup V_r$ such that $|V_i| = |X_i| = n$ and (V_i, V_j) is (ε, d')-super-regular for $d' \ge d$. Then we can embed H into G such that $\phi(X_i) = V_i$.*

Remark 2.6.

On the proof of the Blow-up lemma,

- *the proof of the embedding lemma (greedy embedding) can embed an ε-proportion of vertices.*
- *a careful randomized embedding (random greedy embedding) can embed an $(1-\varepsilon)$-proportion of vertices, succeeding with high probability.*
- *if we run the randomized embedding carefully, we can apply Hall-type result for the remaining vertices and obtain full embedding (Blow-up lemma).*

Exercise 2.7. *Suppose $\varepsilon \ll d \le 1$. If (A, B) is (ε, d)-regular in G, then there exist $A' \subseteq A$, $B' \subseteq B$ such that $|A'| \ge (1-\varepsilon)|A|$, $|B'| \ge (1-\varepsilon)|B|$ and (A', B') is $(2\varepsilon, d)$-super-regular in G.*

Moving to hypergraph regularity

One of the main motivation of the hypergraph regularity is to understand/derive the hypergraph removal lemma. Recall that for $k \geq 2$, a *k-uniform hypergraph* H is a pair of (V, E), where V is a vertex set and E is a family of k-element subsets of V. For the convenience, we usually use *k-graph* to denote the k-uniform hypergraph.

Lemma 2.8 (Hypergraph removal lemma). *For every r-graph H and $\varepsilon > 0$, there exists $\delta > 0$ such that every n-vertex r-graph with $< \delta n^{v(H)}$ copies of H can be made H-free by removing $< \varepsilon n^r$ edges.*

Next we will give the following result, which is a corollary of the tetrahedron removal lemma.

Corollary 2.9. *If G is a 3-graph such that every edge is contained in a unique tetrahedron (i.e., a clique on four vertices), then $e(G) = o(n^3)$.*

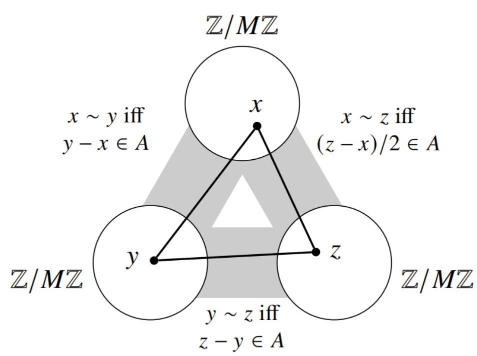
Now let's prove Roth's theorem and Szemerédi's Theorem for 4-AP. We write k-AP for k-term arithmetic progression. We say that A is *3-AP-free* if there are no $x, x+y, x+2y \in A$ with $y \neq 0$.

Theorem 2.10 (Roth's theorem). *Let $A \subseteq [N]$ be 3-AP-free. Then $|A| = o(N)$.*

Proof. Embed $A \subseteq \mathbb{Z}/M\mathbb{Z}$ with $M = 2N + 1$ (to avoid wraparounds). Since A is 3-AP-free in $\mathbb{Z}$, it is 3-AP-free in $\mathbb{Z}/M\mathbb{Z}$ as well.

Now, we construct a tripartite graph G whose parts X, Y, Z are all copies of $\mathbb{Z}/M\mathbb{Z}$. The edges of the graph are (since M is odd, we are allowed to divide by 2 in $\mathbb{Z}/M\mathbb{Z}$):

- $(x, y) \in X \times Y$ whenever $y - x \in A$;
- $(y, z) \in Y \times Z$ whenever $z - y \in A$;
- $(x, z) \in X \times Z$ whenever $(z - x)/2 \in A$.



In this graph, $(x, y, z) \in X \times Y \times Z$ is a triangle if and only if

$$y - x, \frac{z - x}{2}, z - y \in A.$$

The graph was designed so that the above three numbers form an arithmetic progression in the listed order. Since A is 3-AP-free, these three numbers must all be equal. So, every edge of G lies in a unique triangle, formed by setting the three numbers above to be equal.

The graph G has exactly $3M = 6N + 3$ vertices and $3M|A|$ edges. As every edge lies in a unique triangle, G has exactly $3M|A|/3 = M|A| = o(M^3)$ triangles, and the triangle removal lemma says that G can be made triangle-free by removing $o(M^2)$ edges. However, as every edge of G is in a unique triangle, removing any edge destroys at most one triangle. That is, to make G triangle-free, one has to remove at least $M|A|$ edges. Therefore, we have $M|A| = o(M^2)$, yielding $|A| = o(M) = o(N)$ and we are done. □

In fact, Roth's theorem is the first case of a famous result known as Szemerédi's theorem.

Theorem 2.11 (Szemerédi's theorem)**.** *For every fixed $k \geq 3$, every k-AP-free subset of $[N]$ has size $o(N)$.*

Proof of Szemerédi's theorem for 4-AP. Let $A \subseteq [N]$ be 4-AP-free. Let $M = 6N + 1$. Then A is also a 4-AP-free subset in $\mathbb{Z}/M\mathbb{Z}$. Build a 4-partite 3-graph G with parts W, X, Y, Z, all of which are copies of $[M]$. Define edges of G as follows, where w, x, y, z range over elements of W, X, Y, Z, respectively:

$$\begin{aligned} wxy \in E(G) &\iff 3w + 2x + y \in A, \\ wxz \in E(G) &\iff 2w + x - z \in A, \\ wyz \in E(G) &\iff w - y - 2z \in A, \\ xyz \in E(G) &\iff -x - 2y - 3z \in A. \end{aligned}$$

What is important here is that the ith expression does not contain the ith variable.

The vertices $xyzw$ form a tetrahedron if and only if

$$3w + 2x + y,\ 2w + x - z,\ w - y - 2z,\ -x - 2y - 3z \in A.$$

However, these values form a 4-AP with common difference $-x - y - z - w$. Since A is 4-AP-free, the only tetrahedra in A are trivial 4-APs (those with common difference zero). For each triple $(w, x, y) \in W \times X \times Y$, there is exactly one $z \in \mathbb{Z}/M\mathbb{Z}$ such that $x + y + z + w = 0$. Thus, every edge of the hypergraph lies in exactly one tetrahedron.

By Corollary 2.9, the number of edges in the hypergraph is $o(M^3)$. On the other hand, the number of edges is exactly $4M^2|A|$ (for example, for every $a \in A$, there are exactly M^2 triples $(w, x, y) \in (\mathbb{Z}/M\mathbb{Z})^3$ with $3w + 2x + y = a$). Therefore $|A| = o(M) = o(N)$. □

Lecture 3 The hypergraph regularity

Definition 3.1 (Weak hypergraph regularity)**.** *Let H be a 3-graph, and $A, B, C \subseteq V(H)$ be mutually disjoint non-empty vertex set. Then the triple (A, B, C) is called (ε, d)-regular if $|X| \geq \varepsilon|A|, |Y| \geq \varepsilon|B|, |Z| \geq \varepsilon|C|$ for all $X \subseteq A, Y \subseteq B, Z \subseteq C$ and $d_H(X, Y, Z) = (1 \pm \varepsilon)d$ (that is, $|e_H(X, Y, Z)| = (1 \pm \varepsilon)d|X||Y||Z|$).*

However, the weak regularity does not guarantee a counting lemma in general, which is the key component in applications, e.g., in the proof of the removal lemmas. This can be seen from the following example.

Example 3.2. *Let V_1, V_2, V_3, V_4 be vertex sets of size n such that (V_i, V_{i+1}, V_{i+2}) is (ε, d_i)-regular for $i = 1, 2$. Let $P = (v_1, v_2, v_3, v_4)$ be the 3-graph with edges $v_1v_2v_3$ and $v_2v_3v_4$. Then the number of $P = (v_1, v_2, v_3, v_4)$ with $v_i \in V_i$ is not necessarily $(1 \pm o(1))d_1d_2n^4$.*

A k-graph F is linear if $|e \cap e'| \leq 1$ for all $e, e' \in E(F)$.

Remark 3.3.

- *Kohayakawa, Nagle, Rödl, Schacht proved that weak regularity guarantees F-counting iff F is a linear k-graph.*

This motivates us to consider stronger regularity notions that (at least) guarantees counting (which then would suffice for removal lemma).

(Strong) hypergraph regularity

For cleaner notation, let us restrict our discussions to 3-graphs. Let (i, j)-graph denote the j-partite i-graph. To better describe the partite structure, we need the following notation.

- **$\mathcal{P}$-partite hypergraph:** Let $\mathcal{P} = (V_1, V_2, \ldots, V_s)$ be a partition of V. A set $S \subseteq V$ is *$\mathcal{P}$-partite* if $|S \cap V_i| \leq 1$ for all $i = 1, \ldots, s$. A hypergraph is *$\mathcal{P}$-partite* if all of its edges are $\mathcal{P}$-partite. It is *S-partite* if it is $\mathcal{P}$-partite for some $|\mathcal{P}| = S$.

- **Complex:** The complex is a hypergraph H such that if $e \in E(H)$ and $e' \subset e$ with $e' \neq \varnothing$, then $e' \in E(H)$. A 3-complex is a hypergraph H such that if $e \in E(H)$ with $|e| \leq 3$ and $e' \subset e$ with $e' \neq \varnothing$, then $e' \in E(H)$. Let H be a $\mathcal{P}$-partite 3-complex. For $i \leq 3, X \in \binom{\mathcal{P}}{i}$, we write H_X for the subgraph of H_i induced by $\cup X$. For example, if $X = \{V_1, V_2, V_3\}$, then $H_X = H_{\{V_1,V_2,V_3\}} = H_3[V_1 \cup V_2 \cup V_3]$. Next we use $H_{X<}$ to denote the downward closure of H_X but then with edges of H_X itself removed. Then $H_{X<}$ is a $(i-1, i)$-complex. For example, if $X = \{V_1, V_2, V_3\}$ and H_X is a (3,3)-graph, then $H_{X<}$ is a $(2, 3)$-complex. In fact, if $e \in H_X$, then $e' \in H_{X<}$ for all $e' \subsetneq e$ and $e' \neq \varnothing$.

- **Relative density:** $\frac{\textbf{the number of 3-edges in } H_X}{\textbf{the number of triangles in } H_{X<}}$.

  Let H_i be an (i,i)-graph and H_{i-1} be an $(i-1,i)$-graph on the same partition $\mathcal{P}$. Let $K_i(H_{i-1})$ be a family of $\mathcal{P}$-partite i-sets forming a copy of complete $(i-1)$-graph in H_{i-1}. Then the density of H_i with respect to H_{i-1} is defined as follows:

$$d(H_i|H_{i-1}) = \begin{cases} \frac{|K_i(H_{i-1}) \cap E(H_i)|}{|K_i(H_{i-1})|} & \text{if } |K_i(H_{i-1})| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

  For example, for partition $\mathcal{P} = (V_1, V_2, V_3)$, if H_2 is a $(2,3)$-graph and H_3 is a $(3,3)$-graph, then

$$d(H_3|H_2) = \frac{|K_3(H_2) \cap E(H_3)|}{|K_3(H_2)|} = \frac{\text{triangle} \cap \text{3-edges}}{\text{the number of triangles}}.$$

  More generally, if $\overrightarrow{Q} = (Q_1, \dots, Q_r)$ is a collection of r subhypergraphs of H_{i-1}, then we define $K_i(\overrightarrow{Q}) = \bigcup_{j=1}^{r} K_i(Q_j)$ and

$$d(H_i|\overrightarrow{Q}) = \begin{cases} \frac{|K_i(\overrightarrow{Q}) \cap E(H_i)|}{|K_i(\overrightarrow{Q})|} & \text{if } |K_i(\overrightarrow{Q})| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.4 (i-th level is regular with respect to the $(i-1)$-th level)**.** *An (i,i)-graph H_i is (d_i, ε, r)-*regular *with respect to H_{i-1} if for all r-tuples $\overrightarrow{Q}$ with $|K_i(\overrightarrow{Q})| > \varepsilon |K_i(H_{i-1})|$, we have $d(H_i|\overrightarrow{Q}) = d_i \pm \varepsilon$.*

Definition 3.5 (Complex regular)**.** *Given $s \geq 2$ and a $(2,s)$-complex H with a partition $\mathcal{P}$, we say that H is (d_2, ε, r)-regular if for all $A \in \binom{\mathcal{P}}{2}$, H_A is (d_2, ε)-regular with respect to $(H_{A<})$.*
Given $s \geq 3$ and a $(3,s)$-complex H with a partition $\mathcal{P}$, we say that H is $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular if

- *for all $A \in \binom{\mathcal{P}}{2}$, H_A is (d_2, ε)-regular with respect to $(H_{A<})$ or $d(H_A|H_{A<}) = 0$,*
- *for all $A \in \binom{\mathcal{P}}{3}$, H_A is (d, ε_3, r)-regular with respect to $(H_{A<})_2$ or $d(H_A|(H_{A<})_2) = 0$.*

Lemma 3.6 (Restriction)**.** *Let $s, r, m \in \mathbb{N}$, $\alpha, d_2, d, \varepsilon, \varepsilon_3 > 0$ such that $\frac{1}{m} \ll \frac{1}{r}$, $\varepsilon \leq \min\{\varepsilon, d_2\} \leq \varepsilon_3 \ll \alpha \ll d, \frac{1}{s}$. Let H be a $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular $(3,s)$-complex with vertex classes $V_1, \dots, V_s$ of size m. For each i, let $V_i' \subseteq V_i$ be a set of size at least αm. Then the restriction $H' = H[V_1' \cup \dots \cup V_s']$ is $(d, d_2, \sqrt{\varepsilon_3}, \sqrt{\varepsilon}, r)$-regular.*

For the following two definitions, suppose that G is a $(3,\ell)$-complex with parts $(V_1, \dots, V_\ell)$ and H is a $(3,\ell)$-complex with parts $(X_1, \dots, X_\ell)$.

Definition 3.7. *We say that G respects the partition of H if whenever H contains an i-edge with vertices in $X_{j_1}, \dots, X_{j_i}$, then there is an i-edge of G with vertices in $V_{j_1}, \dots, V_{j_i}$.*

Definition 3.8. *A labeled copy of H in G is partition-respecting if for all $i \in [\ell]$, the vertices corresponding to those in X_i lie within V_i.*

In general, we denote the number of labeled partition-respecting copies of H in G by $|H|_G$.

Lemma 3.9 (Extension lemma)**.** *Let r, b, b', m_0 be integers, $b' < b$, and let $\frac{1}{m_0} \ll \{\frac{1}{r}, \varepsilon\} \ll c \ll \min\{\varepsilon_3, d_2\} \leq \varepsilon_3 \ll \theta, \frac{1}{\ell}, d, \frac{1}{b}$, with $\frac{1}{d_2} \in \mathbb{N}$. The following holds for $m \geq m_0$: Suppose that G is a $(3,\ell)$-complex on b vertices with classes $X_1, \ldots, X_\ell$, and let G' be an induced subcomplex of G on b' vertices. Suppose H^* is a $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular $(3,\ell)$-complex with vertex classes $V_1, \ldots, V_\ell$, each of order m, which respects the partition of G. Then all but at most $\theta|G'|_{H^*}$ labeled partition-respecting copies of G' in H^* can extend to at least $cm^{b-b'}$ labeled partition-respecting copies of G in H^*.*

Remark 3.10 (Counting)**.**

- *If $b' = 0$, one has counting: H^* contains $(1 \pm \varepsilon) d^{|e(G_3)|} d_2^{|e(G_2)|} m^b = cm^b$ labeled partition-respecting copies of G.*
- *One can think of the extensions as the "rooted-counting", that is, counting copies of G with prescribed vertices (root vertices).*

Lecture 4 The hypergraph regularity lemma and its applications

Before giving the regularity lemma of Rödl and Schacht, we introduce some notations. Let V be the vertex set and $\mathcal{P}^{(1)} = (V_1, \ldots, V_t)$ be a partition of V, where V_i is cluster for $i \in [t]$.

Definition 4.1. *For all $j \in [3]$, let $Cross_j = Cross_j(P^{(1)})$ denote the set of all crossing j-subsets of V. For all $A \subseteq [t]$, let $Cross_A$ denote all crossing subsets of V that meet V_i if and only if $i \in A$.*

Suppose that $\mathcal{P}_A$ is a partition of Cross_A, where the parts are called cells and $\mathcal{P}^{(2)}$ is the union of all $\mathcal{P}_A$ with $|A| = 2$ (so $\mathcal{P}^{(2)}$ partitions Cross_2).

Definition 4.2. *Given $\mathcal{P} = \{P^{(1)}, P^{(2)}\}$, a family of partitions on V, and $K = v_i v_j v_k$ with $v_i \in V_i$, $v_j \in V_j$, $v_k \in V_k$, the* polyad (or triad) *$\mathcal{P}(K)$ is a $(2,3)$-graph (i.e., 3-partite 2-graph) on $V_i \cup V_j \cup V_k$ with edge set $C(v_i, v_j) \cup C(v_i, v_k) \cup C(v_j, v_k)$ where $C(v_i, v_j)$ is the cell in $\mathcal{P}_{ij}$ that contains $v_i v_j$.*

We say that $\mathcal{P}(K)$ is called (d_2, δ)-regular if all $C(v_i, v_j), C(v_i, v_k), C(v_j, v_k)$ are (d_2, δ)-regular with respect to their underlying sets. Let $\hat{\mathcal{P}}^{(2)}$ be the family of all $\mathcal{P}(K)$ for $K \in \mathrm{Cross}_3$.

Lemma 4.3 (Regularity lemma, Rödl-Schacht, similar to Frankl-Rödl)**.** *For all $\varepsilon_3 > 0$, $t_0 \in \mathbb{N}$ and functions $r : \mathbb{N} \to \mathbb{N}$ and $\varepsilon : \mathbb{N} \to (0,1]$, there exists $d_2 > 0$ such that $\frac{1}{d_2} \in \mathbb{N}$ and $T, n_0 \in \mathbb{N}$ such that $\frac{1}{d_2} \le T$ and $n \ge n_0$ and $T!|n$, and the following holds. Let H be a 3-graph of order n. Then there exists $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ of V such that*

- *$\mathcal{P}^{(1)} = \{V_1, \ldots, V_t\}$ is a partition of V into t clusters of equal size, $t_0 \le t \le T$.*
- *$\mathcal{P}^{(2)}$ partitions $Cross_2$ into at most T cells.*
- *for all $K \in Cross_3$, $\mathcal{P}(K)$ is $(d_2, \varepsilon(T))$-regular.*
- *H is $(\cdot, \varepsilon_3, r)$-regular with respect to all but at most $\varepsilon_3 t^3 (\frac{1}{d_2})^3$ polyads, i.e., members of $\hat{\mathcal{P}}^{(2)}$.*

Next we will present two important applications of hypergraph regularity, of which the graph version we have proved in the first two lectures.

Application 1: the F-removal lemma.

Theorem 4.4 (F-removal lemma)**.** *Let F be a 3-graph on b vertices and $\alpha > 0$. Then there exists $\delta = \delta(\alpha) > 0$ such that the following holds. If a 3-graph H with n vertices has less than δn^b copies of F, then H can be made F-free by removing less than αn^3 edges.*

Proof. Given a 3-graph F with vertex set $[b]$. We start with choosing the following constants:

$$\frac{1}{n} \ll \frac{1}{m_0} \ll \{\frac{1}{r}, \varepsilon\} \ll c \ll \min\{\varepsilon_3, d_2\} \le \varepsilon_3, \frac{1}{t_0} \ll d, \frac{1}{b}.$$

We further take $\alpha = 2d$ and $\delta = c/(2T^b)$. Let H be an n-vertex 3-graph, and we show that either H can be made F-free by removing αn^3 edges, or it has δn^b copies of F.

We apply the regularity lemma to H with input parameter t_0 and ε_3, and with possibly at most $(T! - 1)$ vertices removed, and obtain a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$, where $\mathcal{P}^{(1)} = \{V_1, V_2, \dots, V_t\}$ and $\mathcal{P}^{(2)}$ is a partition of Cross_2.

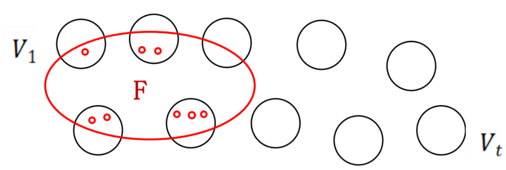
We now proceed the clean step: we remove an edge e from H if

- $e \in E(H)$ not supported on any polyad (as $t \geq t_0$ is large, there are at most $T!n^2 + (n/t)^2 n \leq n^3/t_0$ such edges).

- $e \in E(H)$ supported on a polyad P, but H is not regular with respect to $\mathcal{P}$, (so the number of edges is $\leq \varepsilon_3 t^3 (\frac{1}{d_2})^3 \cdot (d_2^3 + O(\varepsilon_3)) \cdot (\frac{n}{t})^3 \cdot 1 = 2\varepsilon_3 n^3$ by combining the counting lemma for graphs).

- $e \in E(H)$ supported on a polyad P, and H is (d', ε_3, r)-regular with respect to $\mathcal{P}$, but $d' < d$ (so the number of edges is $\leq \binom{t}{3} \cdot (\frac{1}{d_2})^3 \cdot (d_2^3 + O(\varepsilon_3)) \cdot (\frac{n}{t})^3 \cdot d \leq dn^3$).

Altogether, as $1/t_0, \varepsilon_3 \ll d$, we removed at most $2dn^3$ edges of H. Let H' be the resulting graph after deleting these edges of H. Does H' contain a copy of F?



- If no, then we are done.

- If yes, then H' contains a copy of F, and this copy of F defines a $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular complex H^* (by taking the cells that intersect the shadow of F), and H^* respects the partition of F. By Extension/Counting lemma, we derive that H^* has $\geq c(\frac{n-T!}{t})^b \geq (c/2T^b)n^b$ copies of F, and we are also done. □

Application 2: Bounded-degree 3-graphs have linear Ramsey number. (Cooley-Fountoulakis-Kühn-Osthus, Nagle-Olsen-Rödl-Schacht)

Similar to the proof of the graph case, we need to define the reduced 3-graph. However, this definition is indeed not unique and quite depends on the context (the problem). For our Ramsey-type problem, we use the following definition.

Definition 4.5 (Fruitful). *A triple of clusters V_1, V_2, V_3 is fruitful if G is (ε_3, r)-regular with respect to all but $\leq \sqrt{\varepsilon_3}$-fraction of all polyads $\hat{\mathcal{P}}^{(2)}$ induced on V_1, V_2, V_3. Define R to be the reduced 3-graph with vertices $\{V_1, \dots, V_t\}$ and edges as the fruitful triples.*

Lemma 4.6. *All but $\leq 2\sqrt{\varepsilon_3} a_1^3$ of the triples of clusters are fruitful.*

To complete the proof of Ramsey number problem, we use the following two lemmas with $j = 3$.

Lemma 4.7 (Embedding lemma for hypergraphs). *Let Δ, ℓ, r, n_0 be positive integers with $3 \leq \ell$ and let $c, d, d_2, \varepsilon, \varepsilon_3$ be positive constants such that $1/d, 1/d_2 \in \mathbb{N}, 1/n_0 \ll 1/r, \varepsilon \ll \min\{\varepsilon_3, d\} \leq \varepsilon_3 \ll d_2, 1/\Delta, 1/\ell$ and $c \ll d, d_2, 1/\Delta, 1/\ell$. Then the following holds for all integers $n \geq n_0$. Suppose that H is an ℓ-partite 3-uniform hypergraph of maximum degree at most Δ with vertex classes $X_1, \dots, X_\ell$ such that $|X_i| \leq cn$ for all $i = 1, \dots, \ell$. Suppose that for each $i = 2, 3$, $\mathcal{G}_i$ is an ℓ-partite i-uniform hypergraph with vertex classes $V_1, \dots, V_\ell$, which all have size n. Suppose also that $\mathcal{G}_3$ is (d_2, ε_3, r)-regular with respect to $\mathcal{G}_2$, that $\mathcal{G}_2$ is (d_2, ε)-regular, and that $(\mathcal{G}_3, \mathcal{G}_2)$ respects the partition of $\mathcal{H}$. Then $\mathcal{G}_3$ contains a copy of $\mathcal{H}$.*

Lemma 4.8 (Slicing lemma). *Let $j \geq 2$ and $s_0, r \geq 1$ be integers and let δ_0, d_0 and p_0 be positive real numbers. Then there is an integer $n_0 = n_0(j, s_0, r, \delta_0, d_0, p_0)$ such that the following holds. Let $n \geq n_0$ and let $\mathcal{G}_j$ be a j-partite j-uniform hypergraph with vertex classes $V_1, \dots, V_j$ which all have size n. Also let $\mathcal{G}_{j-1}$ be a j-partite $(j-1)$-uniform hypergraph with the same vertex classes and assume that each j-set of vertices that spans a hyperedge in $\mathcal{G}_j$ also spans a $K_{j-1}^{(j-1)}$ in $\mathcal{G}_{j-1}$. Suppose that*

1. *$|\mathcal{K}_j(\mathcal{G}_j)| > n^j / \ln n$ and*

2. *$\mathcal{G}_j$ is (d, δ, r)-regular with respect to $\mathcal{G}_{j-1}$, where $d \geq d_0 \geq 2\delta \geq 2\delta_0$.*

Then for any positive integer $s \leq s_0$ and all positive reals $p_1, \dots, p_s \geq 0$ with $\sum_{i=1}^s p_i \leq 1$ there exists a partition of $E(\mathcal{G}_j)$ into $s+1$ parts $E^{(0)}(\mathcal{G}_j), E^{(1)}(\mathcal{G}_j), \dots, E^{(s)}(\mathcal{G}_j)$ such that if $\mathcal{G}_j(i)$ denotes the spanning subhypergraph of $\mathcal{G}_j$ whose edge set is $E^{(i)}(\mathcal{G}_j)$, then $\mathcal{G}_j(i)$ is $(p_i d, 3\delta, r)$-regular with respect to $\mathcal{G}_{j-1}$ for every $i = 1, \dots, s$. Moreover, $\mathcal{G}_j(0)$ is $((1 - \sum_{i=1}^s p_i)d, 3\delta, r)$-regular with respect to $\mathcal{G}_{j-1}$ and $E^{(0)}(\mathcal{G}_j) = \varnothing$ if $\sum_{i=1}^s p_i = 1$.

The *hypergraph Ramsey number* $R(\mathcal{H})$ of a k-grpah $\mathcal{H}$ is the smallest $n \in \mathbb{N}$ such that for every 2-colouring of the hyperedges of the complete k-graph on n vertices one can find a monochromatic copy of $\mathcal{H}$. The *maximum degree* of $\mathcal{H}$ is the maximum number of hyperedges containing any vertex in $\mathcal{H}$.

Theorem 4.9. *For all Δ, there exists a constant $C = C(\Delta)$ such that all 3-graphs $\mathcal{H}$ of maximum degree at most Δ satisfy $R(\mathcal{H}) \leq C|\mathcal{H}|$.*

Proof. Given Δ, choose large constant C. Consider complete 3-graph $K_m^{(3)}$, $m = C|H|$. Given a red/blue coloring of $E(K_m^{(3)})$. Let G_{red} be the red subgraph and assume that $e(G_{\text{red}}) \geq \frac{1}{2}\binom{m}{3}$. Apply Regularity lemma to G_{red} with $\varepsilon_3 \ll \frac{1}{\Delta}$, obtaining a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$, where $\mathcal{P}^{(1)} = \{V_1, \ldots, V_t\}$ and t is large (e.g., $t \geq \ell := R(K_{3\Delta}^{(3)})$).

Let R be the reduced hypergraph. By Lemma 4.6, we have

$$e(R) \geq (1 - o(1))\binom{|R|}{3} > \left(1 - \frac{1}{\binom{\ell}{3}}\right)\binom{|R|}{3}.$$

Now we claim that R contains a copy of $K_\ell^{(3)}$. Assume for the sake of contradiction that R is $K_\ell^{(3)}$-free. Then for each ℓ-subset S of $V(R)$, we have $e(R[S]) \leq \binom{\ell}{3} - 1$. But note that $e(R) = \binom{|R|-3}{\ell-3}^{-1} \sum_{S \subset V(R), |S| = \ell} e(R[S])$. Thus, we have $e(R) \leq \binom{|R|-3}{\ell-3}^{-1}\binom{|R|}{\ell}\left(\binom{\ell}{3} - 1\right)$. Observe that $\binom{|R|-3}{\ell-3}^{-1}\binom{|R|}{\ell}\binom{\ell}{3} = \binom{|R|}{3}$, which yields the desired contradiction. Without loss of generality, assume it's on $V_1, \ldots, V_\ell$. Choose a $(2, \ell)$-complex S on $V_1, \ldots, V_\ell$ such that S is a union of cells of $\mathcal{P}^{(2)}$ and G_{red} is regular with respect to S. For each $i, j \in [\ell]$, choose a cell on $V_i \times V_j$ uniformly at random ($\frac{1}{d_2}$ choices).

Fix V_i, V_j, V_k with $i, j, k \in [\ell]$, as $V_iV_jV_k \in E(R)$, it is fruitful, G_{red} is (ε_3, r)-regular with respect to $\geq (1 - \sqrt{\varepsilon_3})\left(\frac{1}{d_2}\right)^3$ of the polyads on V_i, V_j, V_k. As we choose each cell uniformly at random, the probability that G_{red} is regular with respect to $S[V_i, V_j, V_k]$ is $\geq 1 - \sqrt{\varepsilon_3}$, and G_{red} is regular with respect to S is $\geq 1 - \sqrt{\varepsilon_3}\binom{\ell}{3} > \frac{1}{2}$ as $\varepsilon \ll \frac{1}{\ell}$. Now color hyperedge $V_iV_jV_k$ red if $d(G_{\text{red}}/S[V_i, V_j, V_k]) \geq \frac{1}{2}$, and color blue otherwise. Since $\ell = R(K_{3\Delta}^{(3)})$, we find a monochromatic copy K of $K_{3\Delta}^{(3)}$ in $K_\ell^{(3)}$. Fix a (3Δ)-vertex-coloring of H such that for each edge of H, all vertices get distinct colors (exists by greedy coloring). First suppose it is red, we want to embed H as a $(3, 3\Delta)$-graph to K. By assumption, G_{red} is $(\cdot, \varepsilon_3, r)$-regular with respect to each polyad of K, and it is partition-respecting with respect to H as it is complete.

The only issue is that the densities are different, which can be dealt with by a simple probabilistic argument (by Slicing lemma). By the embedding lemma, find a copy of H in G_{red}. On the other hand, if K is blue, we need to prove that G_{blue} is regular with respect to all chosen polyads S. So suppose $Q = (Q^{(1)}, \ldots, Q^{(r)})$ is an r-tuple of subhypergraphs of one of these polyads S, satisfying $|K_3(Q)| > \varepsilon_3 |K_3(S)|$. Let d_s be such that G_{red} is (d_s, ε_3, r)-regular with respect to S. Then

$$|(1 - d_s) - d(G_{\text{blue}} \mid Q)| = |d_s - (1 - d(G_{\text{blue}} \mid Q))| = |d_s - d(G_{\text{red}} \mid Q)| < \varepsilon_3.$$

Thus G_{blue} is $(1 - d_s, \varepsilon_3, r)$-regular with respect to S (note that $\varepsilon_3 < \frac{1}{2} < 1 - d_s$). Following the same argument as in the previous case, we add $E(G'_{\text{blue}}) \cap K_3(S)$ to the subcomplex of S induced by the clusters in K to derive the regular $(3, 3\Delta)$-complex S_{blue} to which we can apply the embedding theorem to obtain a copy of H in G_{blue}.

It remains to check that we can choose C to be a constant depending only on Δ. Note that the constants and functions d', ε_3, r, and θ we defined at the beginning of the proof all depend only on Δ. So this is also true for the integers n_0 and t. Note that in order to apply the regularity lemma to G_{red}, we need $m \geq n_0$, where $m = C|H|$. This is certainly true if we set $C \geq n_0$. The embedding theorem allows us to embed subcomplexes of size at most cn, where n is the cluster size and where c satisfies $c \ll \frac{1}{a_1}, \varepsilon_3, \frac{1}{(3\Delta)}$. Thus c too depends only on Δ. In order to apply the embedding theorem, we need that $n \geq n_0$, where n_0 as defined in the embedding theorem depends only on Δ and k. Since the number of clusters is at most t, this is satisfied if $m \geq tn_0$, which in turn is certainly true if $C \geq tn_0$. When we applied the embedding lemma to H, we needed that $|H| \leq cn$. Since

$$n = \frac{m}{a_1} = \frac{C|H|}{a_1} \geq \frac{C|H|}{t},$$

it suffices to choose $C \geq \frac{t}{c}$ for this. Altogether, this shows that we can define the constant C in Theorem 4.9 by

$$C := \max\{tn_0, t/c\}. \qquad \square$$

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