2025 BUPT Summer School - Course III
Regularity Methods and its Applications

Lecturer: Jie Han
Notes prepared by Huan Xu and Jingwen Zhao

Lecture 4 The hypergraph regularity lemma and its applications

Before giving the regularity lemma of Rödl and Schacht, we introduce some notations. Let V be the vertex set and $\mathcal{P}^{(1)} = (V_1, \dots, V_t)$ be a partition of V, where V_i is cluster for $i \in [t]$.

Definition 4.1. *For all $j \in [3]$, let $Cross_j = Cross_j(P^{(1)})$ denote the set of all crossing j-subsets of V. For all $A \subseteq [t]$, let $Cross_A$ denote all crossing subsets of V that meet V_i if and only if $i \in A$.*

Suppose that $\mathcal{P}_A$ is a partition of Cross_A, where the parts are called cells and $\mathcal{P}^{(2)}$ is the union of all $\mathcal{P}_A$ with $|A| = 2$ (so $\mathcal{P}^{(2)}$ partitions Cross_2).

Definition 4.2. *Given $\mathcal{P} = \{P^{(1)}, P^{(2)}\}$, a family of partitions on V, and $K = v_i v_j v_k$ with $v_i \in V_i$, $v_j \in V_j$, $v_k \in V_k$, the* polyad (or triad) *$\mathcal{P}(K)$ is a $(2,3)$-graph (i.e., 3-partite 2-graph) on $V_i \cup V_j \cup V_k$ with edge set $C(v_i, v_j) \cup C(v_i, v_k) \cup C(v_j, v_k)$ where $C(v_i, v_j)$ is the cell in $\mathcal{P}_{ij}$ that contains $v_i v_j$.*

We say that $\mathcal{P}(K)$ is called (d_2, δ)-regular if all $C(v_i, v_j), C(v_i, v_k), C(v_j, v_k)$ are (d_2, δ)-regular with respect to their underlying sets. Let $\hat{\mathcal{P}}^{(2)}$ be the family of all $\mathcal{P}(K)$ for $K \in \text{Cross}_3$.

Lemma 4.3 (Regularity lemma, Rödl-Schacht, similar to Frankl-Rödl)**.** *For all $\varepsilon_3 > 0$, $t_0 \in \mathbb{N}$ and functions $r : \mathbb{N} \to \mathbb{N}$ and $\varepsilon : \mathbb{N} \to (0, 1]$, there exists $d_2 > 0$ such that $\frac{1}{d_2} \in \mathbb{N}$ and $T, n_0 \in \mathbb{N}$ such that $\frac{1}{d_2} \le T$ and $n \ge n_0$ and $T! | n$, and the following holds. Let H be a 3-graph of order n. Then there exists $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ of V such that*

- *$\mathcal{P}^{(1)} = \{V_1, \dots, V_t\}$ is a partition of V into t clusters of equal size, $t_0 \le t \le T$.*
- *$\mathcal{P}^{(2)}$ partitions $Cross_2$ into at most T cells.*
- *for all $K \in Cross_3$, $\mathcal{P}(K)$ is $(d_2, \varepsilon(T))$-regular.*
- *H is $(\cdot, \varepsilon_3, r)$-regular with respect to all but at most $\varepsilon_3 t^3 (\frac{1}{d_2})^3$ polyads, i.e., members of $\hat{\mathcal{P}}^{(2)}$.*

Next we will present two important applications of hypergraph regularity, of which the graph version we have proved in the first two lectures.

Application 1: the F-removal lemma.

Theorem 4.4 (F-removal lemma)**.** *Let F be a 3-graph on b vertices and $\alpha > 0$. Then there exists $\delta = \delta(\alpha) > 0$ such that the following holds. If a 3-graph H with n vertices has less than δn^b copies of F, then H can be made F-free by removing less than αn^3 edges.*

Proof. Given a 3-graph F with vertex set $[b]$. We start with choosing the following constants:

$$\frac{1}{n} \ll \frac{1}{m_0} \ll \{\frac{1}{r}, \varepsilon\} \ll c \ll \min\{\varepsilon_3, d_2\} \le \varepsilon_3, \frac{1}{t_0} \ll d, \frac{1}{b}.$$

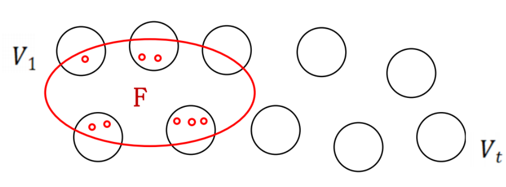
We further take $\alpha = 2d$ and $\delta = c/(2T^b)$. Let H be an n-vertex 3-graph, and we show that either H can be made F-free by removing αn^3 edges, or it has δn^b copies of F.

We apply the regularity lemma to H with input parameter t_0 and ε_3, and with possibly at most $(T! - 1)$ vertices removed, and obtain a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$, where $\mathcal{P}^{(1)} = \{V_1, V_2, \ldots, V_t\}$ and $\mathcal{P}^{(2)}$ is a partition of Cross_2.

We now proceed the clean step: we remove an edge e from H if

- $e \in E(H)$ not supported on any polyad (as $t \ge t_0$ is large, there are at most $T! n^2 + (n/t)^2 n \le n^3/t_0$ such edges).
- $e \in E(H)$ supported on a polyad P, but H is not regular with respect to $\mathcal{P}$, (so the number of edges is $\le \varepsilon_3 t^3 (\frac{1}{d_2})^3 \cdot (d_2^3 + O(\varepsilon_3)) \cdot (\frac{n}{t})^3 \cdot 1 = 2\varepsilon_3 n^3$ by combining the counting lemma for graphs).
- $e \in E(H)$ supported on a polyad P, and H is (d', ε_3, r)-regular with respect to $\mathcal{P}$, but $d' < d$ (so the number of edges is $\le \binom{t}{3} \cdot (\frac{1}{d_2})^3 \cdot (d_2^3 + O(\varepsilon_3)) \cdot (\frac{n}{t})^3 \cdot d \le dn^3$).

Altogether, as $1/t_0, \varepsilon_3 \ll d$, we removed at most $2dn^3$ edges of H. Let H' be the resulting graph after deleting these edges of H. Does H' contain a copy of F?



- If no, then we are done.
- If yes, then H' contains a copy of F, and this copy of F defines a $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular complex H^* (by taking the cells that intersect the shadow of F), and H^* respects the partition of F. By

Extension/Counting lemma, we derive that H^* has $\geq c(\frac{n-T!}{t})^b \geq (c/2T^b)n^b$ copies of F, and we are also done. $\square$

Application 2: Bounded-degree 3-graphs have linear Ramsey number. (Cooley-Fountoulakis-Kühn-Osthus, Nagle-Olsen-Rödl-Schacht)

Similar to the proof of the graph case, we need to define the reduced 3-graph. However, this definition is indeed not unique and quite depends on the context (the problem). For our Ramsey-type problem, we use the following definition.

Definition 4.5 (Fruitful). *A triple of clusters V_1, V_2, V_3 is fruitful if G is (ε_3, r)-regular with respect to all but $\leq \sqrt{\varepsilon_3}$-fraction of all polyads $\hat{\mathcal{P}}^{(2)}$ induced on V_1, V_2, V_3. Define R to be the reduced 3-graph with vertices $\{V_1, \dots, V_t\}$ and edges as the fruitful triples.*

Lemma 4.6. *All but $\leq 2\sqrt{\varepsilon_3} a_1^3$ of the triples of clusters are fruitful.*

To complete the proof of Ramsey number problem, we use the following two lemmas with $j = 3$.

Lemma 4.7 (Embedding lemma for hypergraphs). *Let Δ, ℓ, r, n_0 be positive integers with $3 \leq \ell$ and let $c, d, d_2, \varepsilon, \varepsilon_3$ be positive constants such that $1/d, 1/d_2 \in \mathbb{N}, 1/n_0 \ll 1/r, \varepsilon \ll \min\{\varepsilon_3, d\} \leq \varepsilon_3 \ll d_2, 1/\Delta, 1/\ell$ and $c \ll d, d_2, 1/\Delta, 1/\ell$. Then the following holds for all integers $n \geq n_0$. Suppose that H is an ℓ-partite 3-uniform hypergraph of maximum degree at most Δ with vertex classes $X_1, \dots, X_\ell$ such that $|X_i| \leq cn$ for all $i = 1, \dots, \ell$. Suppose that for each $i = 2, 3$, $\mathcal{G}_i$ is an ℓ-partite i-uniform hypergraph with vertex classes $V_1, \dots, V_\ell$, which all have size n. Suppose also that $\mathcal{G}_3$ is (d_2, ε_3, r)-regular with respect to $\mathcal{G}_2$, that $\mathcal{G}_2$ is (d_2, ε)-regular, and that $(\mathcal{G}_3, \mathcal{G}_2)$ respects the partition of $\mathcal{H}$. Then $\mathcal{G}_3$ contains a copy of $\mathcal{H}$.*

Lemma 4.8 (Slicing lemma). *Let $j \geq 2$ and $s_0, r \geq 1$ be integers and let δ_0, d_0 and p_0 be positive real numbers. Then there is an integer $n_0 = n_0(j, s_0, r, \delta_0, d_0, p_0)$ such that the following holds. Let $n \geq n_0$ and let $\mathcal{G}_j$ be a j-partite j-uniform hypergraph with vertex classes $V_1, \dots, V_j$ which all have size n. Also let $\mathcal{G}_{j-1}$ be a j-partite $(j-1)$-uniform hypergraph with the same vertex classes and assume that each j-set of vertices that spans a hyperedge in $\mathcal{G}_j$ also spans a $K_{j-1}^{(j-1)}$ in $\mathcal{G}_{j-1}$. Suppose that*

1. *$|\mathcal{K}_j(\mathcal{G}_j)| > n^j/\ln n$ and*
2. *$\mathcal{G}_j$ is (d, δ, r)-regular with respect to $\mathcal{G}_{j-1}$, where $d \geq d_0 \geq 2\delta \geq 2\delta_0$.*

Then for any positive integer $s \leq s_0$ and all positive reals $p_1, \dots, p_s \geq 0$ with $\sum_{i=1}^s p_i \leq 1$ there exists a partition of $E(\mathcal{G}_j)$ into $s+1$ parts $E^{(0)}(\mathcal{G}_j), E^{(1)}(\mathcal{G}_j), \dots, E^{(s)}(\mathcal{G}_j)$ such that if $\mathcal{G}_j(i)$ denotes the spanning subhypergraph of $\mathcal{G}_j$ whose edge set is $E^{(i)}(\mathcal{G}_j)$, then $\mathcal{G}_j(i)$ is $(p_i d, 3\delta, r)$-regular with respect to $\mathcal{G}_{j-1}$ for every $i = 1, \dots, s$. Moreover, $\mathcal{G}_j(0)$ is $((1 - \sum_{i=1}^s p_i)d, 3\delta, r)$-regular with respect to $\mathcal{G}_{j-1}$ and $E^{(0)}(\mathcal{G}_j) = \varnothing$ if $\sum_{i=1}^s p_i = 1$.

The *hypergraph Ramsey number* $R(\mathcal{H})$ of a k-grpah $\mathcal{H}$ is the smallest $n \in \mathbb{N}$ such that for every 2-colouring of the hyperedges of the complete k-graph on n vertices one can find a monochromatic copy of $\mathcal{H}$. The *maximum degree* of $\mathcal{H}$ is the maximum number of hyperedges containing any vertex in $\mathcal{H}$.

Theorem 4.9. *For all Δ, there exists a constant $C = C(\Delta)$ such that all 3-graphs $\mathcal{H}$ of maximum degree at most Δ satisfy $R(\mathcal{H}) \le C|\mathcal{H}|$.*

Proof. Given Δ, choose large constant C. Consider complete 3-graph $K_m^{(3)}$, $m = C|H|$. Given a red/blue coloring of $E(K_m^{(3)})$. Let G_{red} be the red subgraph and assume that $e(G_{\text{red}}) \ge \frac{1}{2}\binom{m}{3}$. Apply Regularity lemma to G_{red} with $\varepsilon_3 \ll \frac{1}{\Delta}$, obtaining a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$, where $\mathcal{P}^{(1)} = \{V_1, \dots, V_t\}$ and t is large (e.g., $t \ge \ell := R(K_{3\Delta}^{(3)})$).

Let R be the reduced hypergraph. By Lemma 4.6, we have

$$e(R) \ge (1 - o(1))\binom{|R|}{3} > \left(1 - \frac{1}{\binom{\ell}{3}}\right)\binom{|R|}{3}.$$

Now we claim that R contains a copy of $K_\ell^{(3)}$. Assume for the sake of contradiction that R is $K_\ell^{(3)}$-free. Then for each ℓ-subset S of $V(R)$, we have $e(R[S]) \le \binom{\ell}{3} - 1$. But note that $e(R) = \binom{|R|-3}{\ell-3}^{-1} \sum_{S \subset V(R), |S| = \ell} e(R[S])$. Thus, we have $e(R) \le \binom{|R|-3}{\ell-3}^{-1}\binom{|R|}{\ell}\left(\binom{\ell}{3} - 1\right)$. Observe that $\binom{|R|-3}{\ell-3}^{-1}\binom{|R|}{\ell}\binom{\ell}{3} = \binom{|R|}{3}$, which yields the desired contradiction. Without loss of generality, assume it's on $V_1, \dots, V_\ell$. Choose a $(2, \ell)$-complex S on $V_1, \dots, V_\ell$ such that S is a union of cells of $\mathcal{P}^{(2)}$ and G_{red} is regular with respect to S. For each $i, j \in [\ell]$, choose a cell on $V_i \times V_j$ uniformly at random ($\frac{1}{d_2}$ choices).

Fix V_i, V_j, V_k with $i, j, k \in [\ell]$, as $V_iV_jV_k \in E(R)$, it is fruitful, G_{red} is (ε_3, r)-regular with respect to $\ge (1 - \sqrt{\varepsilon_3})\left(\frac{1}{d_2}\right)^3$ of the polyads on V_i, V_j, V_k. As we choose each cell uniformly at random, the probability that G_{red} is regular with respect to $S[V_i, V_j, V_k]$ is $\ge 1 - \sqrt{\varepsilon_3}$, and G_{red} is regular with respect to S is $\ge 1 - \sqrt{\varepsilon_3}\binom{\ell}{3} > \frac{1}{2}$ as $\varepsilon \ll \frac{1}{\ell}$. Now color hyperedge $V_iV_jV_k$ red if $d(G_{\text{red}}/S[V_i, V_j, V_k]) \ge \frac{1}{2}$, and color blue otherwise. Since $\ell = R(K_{3\Delta}^{(3)})$, we find a monochromatic copy K of $K_{3\Delta}^{(3)}$ in $K_\ell^{(3)}$. Fix a (3Δ)-vertex-coloring of H such that for each edge of H, all vertices get distinct colors (exists by greedy coloring). First suppose it is red, we want to embed H as a $(3, 3\Delta)$-graph to K. By assumption, G_{red} is $(\cdot, \varepsilon_3, r)$-regular with respect to each polyad of K, and it is partition-respecting with respect to H as it is complete.

The only issue is that the densities are different, which can be dealt with by a simple probabilistic argument (by Slicing lemma). By the embedding lemma, find a copy of H in G_{red}. On the other hand, if K is blue, we need to prove that G_{blue} is regular with respect to all chosen polyads S. So suppose $Q = (Q^{(1)}, \dots, Q^{(r)})$ is an r-tuple of subhypergraphs of one of these polyads S, satisfying $|K_3(Q)| > \varepsilon_3 |K_3(S)|$. Let d_s be such that G_{red} is (d_s, ε_3, r)-regular with respect to S. Then

$$|(1 - d_s) - d(G_{\text{blue}} \mid Q)| = |d_s - (1 - d(G_{\text{blue}} \mid Q))| = |d_s - d(G_{\text{red}} \mid Q)| < \varepsilon_3.$$

Thus G_{blue} is $(1-d_s, \varepsilon_3, r)$-regular with respect to S (note that $\varepsilon_3 < \frac{1}{2} < 1-d_s$). Following the same argument as in the previous case, we add $E(G'_{\text{blue}}) \cap K_3(S)$ to the subcomplex of S induced by the clusters in K to derive the regular $(3, 3\Delta)$-complex S_{blue} to which we can apply the embedding theorem to obtain a copy of H in G_{blue}.

It remains to check that we can choose C to be a constant depending only on Δ. Note that the constants and functions d', ε_3, r, and θ we defined at the beginning of the proof all depend only on Δ. So this is also true for the integers n_0 and t. Note that in order to apply the regularity lemma to G_{red}, we need $m \geq n_0$, where $m = C|H|$. This is certainly true if we set $C \geq n_0$. The embedding theorem allows us to embed subcomplexes of size at most cn, where n is the cluster size and where c satisfies $c \ll \frac{1}{a_1}, \varepsilon_3, \frac{1}{(3\Delta)}$. Thus c too depends only on Δ. In order to apply the embedding theorem, we need that $n \geq n_0$, where n_0 as defined in the embedding theorem depends only on Δ and k. Since the number of clusters is at most t, this is satisfied if $m \geq tn_0$, which in turn is certainly true if $C \geq tn_0$. When we applied the embedding lemma to H, we needed that $|H| \leq cn$. Since

$$n = \frac{m}{a_1} = \frac{C|H|}{a_1} \geq \frac{C|H|}{t},$$

it suffices to choose $C \geq \frac{t}{c}$ for this. Altogether, this shows that we can define the constant C in Theorem 4.9 by

$$C := \max\{tn_0, t/c\}. \qquad \square$$

References:

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