2025 BUPT Summer School - Course III
Regularity Methods and its Applications

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Lecture 3 The hypergraph regularity

Definition 3.1 (Weak hypergraph regularity)**.** *Let H be a 3-graph, and $A, B, C \subseteq V(H)$ be mutually disjoint non-empty vertex set. Then the triple (A, B, C) is called (ε, d)-regular if $|X| \geq \varepsilon|A|, |Y| \geq \varepsilon|B|, |Z| \geq \varepsilon|C|$ for all $X \subseteq A, Y \subseteq B, Z \subseteq C$ and $d_H(X, Y, Z) = (1 \pm \varepsilon)d$ (that is, $|e_H(X, Y, Z)| = (1 \pm \varepsilon)d|X||Y||Z|$).*

Example 3.2. *Let V_1, V_2, V_3, V_4 be vertex sets of size n such that (V_i, V_{i+1}, V_{i+2}) is (ε, d_i)-regular for $i = 1, 2$. Let $P = (v_1, v_2, v_3, v_4)$ be the 3-graph with edges $v_1v_2v_3$ and $v_2v_3v_4$. Then the number of $P = (v_1, v_2, v_3, v_4)$ with $v_i \in V_i$ is not necessarily $(1 \pm o(1))d_1d_2n^4$.*

A k-graph F is linear if $|e \cap e'| \leq 1$ for all $e, e' \in E(F)$.

Remark 3.3.

- *Kohayakawa, Nagle, Rödl, Schacht proved the following result:*

$$\text{Weak regularity} \iff \text{Counting linear } k\text{-graphs } F.$$

Hypergraph regularity

Let (i, j)-graph denote the j-partite i-graph.

- **Regular complexes:** Let $\mathcal{P} = (V_1, V_2, \ldots, V_s)$ be a partition of V. A set $S \subseteq V$ is $\mathcal{P}$*-partite* if $|S \cap V_i| \leq 1$ for all $i = 1, \ldots, s$. A hypergraph is $\mathcal{P}$*-partite* if all of its edges are $\mathcal{P}$-partite. It is S*-partite* if it is $\mathcal{P}$-partite for some $|\mathcal{P}| = S$.

- **Complex:** The complex is a hypergraph H such that if $e \in E(H)$ and $e' \subset e$ with $e' \neq \varnothing$, then $e' \in E(H)$. A 3-complex is a hypergraph H such that if $e \in E(H)$ with $|e| \leq 3$ and $e' \subset e$ with $e' \neq \varnothing$, then $e' \in E(H)$. Let H be a $\mathcal{P}$-partite 3-complex. For $i \leq 3, X \in \binom{\mathcal{P}}{i}$, we write H_X for the subgraph of

H_i induced by $\cup X$. For example, if $X = \{V_1, V_2, V_3\}$, then $H_X = H_{\{V_1,V_2,V_3\}} = H_3[V_1 \cup V_2 \cup V_3]$. Next we use $H_{X<}$ to denote H_X downward closure but then remove H_X. Then $H_{X<}$ is a $(i-1,i)$-complex. For example, if $X = \{V_1, V_2, V_3\}$ and H_X is a (3,3)-graph, then $H_{X<}$ is a $(2,3)$-complex. In fact, if $e \in H_X$, then $e' \in H_{X<}$ for all $e' \subsetneq e$ and $e' \neq \varnothing$.

- **Relative density:** $\frac{\textbf{the number of 3-edges in } H_X}{\textbf{the number of triangles in } H_{X<}}$.

  Let H_i be an (i,i)-graph and H_{i-1} be an $(i-1,i)$-graph on the same partition $\mathcal{P}$. Let $K_i(H_{i-1})$ be a family of $\mathcal{P}$-partite i-sets forming a copy of complete $(i-1)$-graph in H_{i-1}. Then the density of H_i with respect to H_{i-1} is as follows:

$$d(H_i|H_{i-1}) = \begin{cases} \frac{|K_i(H_{i-1}) \cap E(H_i)|}{|K_i(H_{i-1})|} & \text{if } |K_i(H_{i-1})| > 0, \\ 0 & \text{otherwise .} \end{cases}$$

  For example, for partition $\mathcal{P} = (V_1, V_2, V_3)$, if H_2 is a $(2,3)$-graph and H_3 is a $(3,3)$-graph, then

$$d(H_3|H_2) = \frac{|K_3(H_2) \cap E(H_3)|}{|K_3(H_2)|} = \frac{\text{triangle} \cap \text{3-edges}}{\text{the number of triangles}}.$$

  More generally, if $\overrightarrow{Q} = (Q_1, \dots, Q_r)$ is a collection of r subhypergraphs of H_{i-1}, then we define $K_i(\overrightarrow{Q}) = \bigcup_{j=1}^r K_i(Q_j)$ and

$$d(H_i|\overrightarrow{Q}) = \begin{cases} \frac{|K_i(\overrightarrow{Q}) \cap E(H_i)|}{|K_i(\overrightarrow{Q})|} & \text{if } |K_i(\overrightarrow{Q})| > 0, \\ 0 & \text{otherwise .} \end{cases}$$

Definition 3.4. *An (i,i)-graph H_i is (d_i, ε, r)-*regular *with respect to H_{i-1} if for all r-tuples $\overrightarrow{Q}$ with $|K_i(\overrightarrow{Q})| > \varepsilon |K_i(H_{i-1})|$, we have $d(H_i|\overrightarrow{Q}) = d_i \pm \varepsilon$.*

Definition 3.5 (Complex regular)**.** *Given $s \geq 2$ and a $(2,s)$-complex H with a partition $\mathcal{P}$, we say that H is (d_2, ε, r)-regular if for all $A \in \binom{\mathcal{P}}{2}$, H_A is (d_2, ε)-regular with respect to $(H_{A<})$.*
Given $s \geq 3$ and a $(3,s)$-complex H with a partition $\mathcal{P}$, we say that H is $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular if

- *for all $A \in \binom{\mathcal{P}}{2}$, H_A is (d_2, ε)-regular with respect to $(H_{A<})$ or $d(H_A|H_{A<}) = 0$,*
- *for all $A \in \binom{\mathcal{P}}{3}$, H_A is (d, ε_3, r)-regular with respect to $(H_{A<})_2$ or $d(H_A|(H_{A<})_2) = 0$.*

Lemma 3.6 (Restriction)**.** *Let $s, r, m \in \mathbb{N}$, $\alpha, d_2, d, \varepsilon, \varepsilon_3 > 0$ such that $\frac{1}{m} \ll \frac{1}{r}$, $\varepsilon \leq \min\{\varepsilon, d_2\} \leq \varepsilon_3 \ll \alpha \ll d, \frac{1}{s}$. Let H be a $(d, d_2, \varepsilon_3, \varepsilon, r)$-regular $(3,s)$-complex with vertex classes $V_1, \dots, V_s$ of size m. For each i, let $V_i' \subseteq V_i$ be a set of size at least αm. Then the restriction $H' = H[V_1' \cup \dots \cup V_s']$ is $(d, d_2, \sqrt{\varepsilon_3}, \sqrt{\varepsilon}, r)$-regular.*

For the following two definitions, suppose that G is a $(3,\ell)$-complex with parts $(V_1, \dots, V_\ell)$ and H is a $(3,\ell)$-complex with parts $(X_1, \dots, X_\ell)$.

Definition 3.7. *We say that G respects the partition of H if whenever H contains an i-edge with vertices in $X_{j_1}, \dots, X_{j_i}$, then there is an i-edge of G with vertices in $V_{j_1}, \dots, V_{j_i}$.*

Definition 3.8. *A labeled copy of H in G is partition-respecting if for all $i \in [\ell]$, the vertices corresponding to those in X_i lie within V_i.*

In general, we denote the number of labeled partition-respecting copies of H in G by $|H|_G$.

Lemma 3.9 (Extension lemma)**.** *Let r, b, b', m_0 be integers, $b' < b$, and let $\frac{1}{m_0} \ll \{\frac{1}{r}, \delta\} \ll c \ll \min\{\delta_3, d_2\} \leq \delta_3 \ll \theta, \frac{1}{r}, d, \frac{1}{b}, \frac{1}{d_2} \in \mathbb{N}$. The following holds for $m \geq m_0$: Suppose that G is a $(3, \ell)$-complex on b vertices with classes $X_1, \dots, X_\ell$, and let G' be an induced subcomplex of G on b' vertices. Suppose H^* is a $(d, d_2, \delta_3, \delta, r)$-regular $(3, \ell)$-complex with vertex classes $V_1, \dots, V_\ell$, each of order m, which respects the partition of G. Then all but at most $\theta |G'|_{H^*}$ labeled partition-respecting copies of G' in H^* can extend to at least $cm^{b-b'}$ labeled partition-respecting copies of G in H^*.*

Remark 3.10 (Counting)**.**

- *If $b' = 0$, one has counting: H^* contains $(1 \pm \varepsilon) d^{e(G_3)} d_2^{e(G_2)} n^b$ labeled partition-respecting copies of G.*
- *Question (Counting): count the number of labeled partition-respecting copies of K_3.*
- *Question (rooted counting): count the number of labeled partition-respecting copies of K_3 containing the root v.*

Before giving the regularity lemma of Rödl and Schacht, we introduce some notations. Let V be the vertex set and $\mathcal{P}^{(1)} = (V_1, \dots, V_t)$ be a partition of V, where V_i is cluster for $i \in [t]$. For all $j \in [3]$, let $\mathrm{Cross}_j = \mathrm{Cross}_j(P^{(1)})$ denote the set of all crossing j-subsets of V. For all $A \subseteq [t]$, let Cross_A denote all crossing subsets of V that meet V_i if and only if $i \in A$. Suppose that $\mathcal{P}_A$ is a partition of Cross_A, where the parts are called cells and $\mathcal{P}^{(2)}$ is the union of all $\mathcal{P}_A$ with $|A| = 2$ (so $\mathcal{P}^{(2)}$ partitions Cross_2). Let $\mathcal{P} = \{P^{(1)}, P^{(2)}\}$ be a family of partitions on V. Given $\mathcal{P} = \{P^{(1)}, P^{(2)}\}$ and $K = v_i v_j v_k$ with $v_i \in V_i$, $v_j \in V_j$, $v_k \in V_k$, the *polyad (or triad)* $\mathcal{P}(K)$ is a $(2,3)$-graph (i.e.,3-partite 2-graph) on $V_i \cup V_j \cup V_k$ with edge set $C(v_i, v_j) \cup C(v_i, v_k) \cup C(v_j, v_k)$ where $C(v_i, v_j)$ is the cell in $\mathcal{P}_{ij}$ that contains $v_i v_j$. $\mathcal{P}(K)$ is called (d_2, δ)-regular if all $C(v_i, v_j), C(v_i, v_k), C(v_j, v_k)$ are (d_2, δ)-regular with respect to their underlying sets. Let $\hat{\mathcal{P}}^{(2)}$ be the family of all $\mathcal{P}(K)$ for $K \in \mathrm{Cross}_3$.

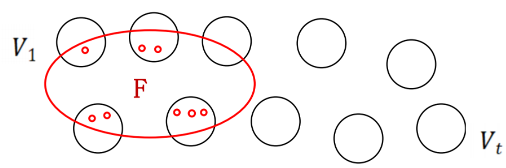
Lemma 3.11 (Regularity lemma-Rödl, Schacht, similar to Frankl-Rödl)**.** *For all* $\varepsilon_3 > 0$, $t_0 \in \mathbb{N}$ *and functions* $\gamma : \mathbb{N} \to \mathbb{N}$ *and* $\xi : \mathbb{N} \to (0,1]$*, there exists* $d_2 > 0$ *such that* $\frac{1}{d_2} \in \mathbb{N}$ *and* $T, n_0 \in \mathbb{N}$ *such that* $\frac{1}{d_2} \leq T$ *and* $n \geq n_0$ *and* $T!|n$*, and the following holds. Let* H *be a 3-graph of order* n*. Then there exists* $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ *of* V *such that*

- $\mathcal{P}^{(1)} = \{V_1, \dots, V_t\}$ *is a partition of* V *into* t *clusters of equal size,* $t_0 \leq t \leq T$*.*
- $\mathcal{P}^{(2)}$ *partitions* $Cross_2$ *into at most* T *cells.*
- *for all* $K \in Cross_3$*,* $\mathcal{P}(K)$ *is* $(d_2, \varepsilon(T))$*-regular.*
- H *is* $(\cdot, \varepsilon_3)$*-regular with respect to all but at most* $\varepsilon_3 t^3 (\frac{1}{d_2})^3$ *polyads, i.e., members of* $\hat{\mathcal{P}}^{(2)}$*.*

Theorem 3.12 (F-removal lemma)**.** *Let* F *be a 3-graph on* b *vertices. If a 3-graph* H *with* n *vertices has* $o(n^b)$ *copies of* F*, then* H *can be made* F*-free by removing at most* $o(n^3)$ *edges.*

Proof.

- Firstly, we choose the desired constants.
- Then apply regularity lemma to H - possibly at most $(T! - 1)$ vertices.
- Define $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ as follows:
  $\mathcal{P}^{(1)}$:

  $\mathcal{P}^{(2)}$: a partition of Cross$_2$.
- $e \in E(H)$ not supported on any polyad is deleted (so the number of edges is $o(n^3)$).
- $e \in E(H)$ not supported on a polyad P, but H is not regular with respect to $\mathcal{P}$, is deleted (so the number of edges is $o(n^3)$).
- Let H' be the obtained graph after deleting $o(n^3)$ edges of H. Does H' contain a copy of F?
  - If No, done!

– If yes, there exists $F \subseteq H'$, then one can apply Counting lemma and get at least $o(n^b)$ copies of F in $H' \subseteq H$, a contradiction.

$\square$