2025 BUPT Summer School - Course III
Regularity Methods and its Applications

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Lecture 2 The regularity method and the blowup lemma

In Lecture 1 we use an embedding scheme for proving the K_3-counting lemma. If we do the embedding a bit more carefully, then we can embed a subgraph of small linear size.

Lemma 2.1 (Graph embedding lemma). *For any $\Delta, n, m \in \mathbb{N}$, let $\varepsilon < \frac{(d-\varepsilon)^\Delta}{\Delta+2}$ and $m \leq \varepsilon n$. Suppose that G is a graph satisfying $v(G) = V_1 \cup \cdots \cup V_r$ with $|V_i| \geq n$ for $i \in [r]$ and (V_i, V_j) is (ε, d')-regular, where $d' \geq d$ and $i \neq j \in [r]$. Let H be an h-vertex r-partite graph with partition $X_1 \cup \cdots \cup X_r$ with maximum degree Δ and $|X_i| \leq m$ for $i \in [r]$. Then $H \subseteq G$.*

Proof. Let $v(H) = \{x_1, \dots, x_h\}$, and let $\phi(i) \in [r]$ such that $x_i \in X_{\phi(i)}$. We will construct an embedding f by defining $V_1 = f(x_1), V_2 = f(x_2), \dots, v_h = f(x_h)$. Denote $C_i(j)$ to be the set of possible candidates of v_j after we determine $v_1, v_2, \dots, v_{i-1}$.

Now we embed X_i to V_i. Initially, $C_i(j) = V_{\phi(j)}$. Suppose that we have determined $v_1, v_2, \dots, v_{i-1}$ and we have $|C_i(j)| \geq (\Delta+1)\varepsilon n$ for all $j \geq i$. Then we need to select v_i from $C_i(i)$. Consider $A = \{x_j \in N_H(x_i) : j > i\} = \{x_{s_1}, \dots, x_{s_p}\}$, $p \leq \Delta$. Since $(V_{\phi(i)}, V_{\phi(s_\ell)})$ is (ε, d')-regular for $\ell \in [p]$, all but at most εn vertices in $C_i(i)$ have at least $(d-\varepsilon)|C_i(s_\ell)|$ neighbors in $C_i(s_\ell)$ for $\ell \in [p]$. Thus, there exist at least $|C_i(i)| - \Delta\varepsilon n$ vertices in $C_i(i)$ have at least $(d-\varepsilon)|C_i(S_\ell)|$ neighbors in $C_i(s_\ell)$ for $\ell \in [p]$. Among them, at most $m-1 \leq \varepsilon n - 1$ vertices are in $\{v_1, v_2, \dots, v_{i-1}\}$. Then we can choose one vertex as $v_i \notin \{v_1, v_2, \dots, v_{i-1}\}$ that has at least $(d-\varepsilon)|C_i(S_\ell)|$ neighbors in $C_i(s_\ell)$ for $\ell \in [p]$.

Let $v_i = f(x_i)$. Next, we will make the following update. Let

$$C_{i+1}(j) = \begin{cases} C_i(j) \cap N_G(v_i) & \text{if } x_j \sim x_i, \\ \\ C_i(j) & \text{if } x_j \nsim x_i. \end{cases}$$

Since x_j has at most Δ neighbors, $|C_{i+1}(j)| \geq n(d-\varepsilon)^\Delta \geq (\Delta+1)\varepsilon n$ throughout the process. Thus, we can always choose v_i for $i \leq h$. Then we obtain an embedding of $H \subseteq G$ satisfying the partition. $\square$

Application: Ramsey Theory

Given a graph H, let $r(H) = \min n$ such that for all edge-coloring of K_n contains a monochromatic copy of H.

Theorem 2.2 (Chvátal - Rödl - Szemerédi - Trotter). *Fix Δ and let H be a graph with $\Delta(H) \le \Delta$. Then there exists $c = c(\Delta)$ such that $r(H) \le c|H|$.*

Proof. Let $k = r(K_{\Delta+1})$. Take $\varepsilon = \frac{1}{2^{\Delta+1}k}$, $t = \Delta + 1$. Assume that $N = N(\varepsilon, t)$, $T = T(\varepsilon, t)$ as defined in regularity lemma. Let $c = c(\Delta) = \max\{3T/\varepsilon, N\}$. Take $n > c|H| = 3T|H|/\varepsilon$. Next, we need to show that every 2-edge-coloring of $E(K_n)$ contains a monochromatic copy of H.

Let G be the red graph. Applying the regularity lemma to G with ε, t, we can obtain an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_r$ for G with $|V_i| \ge (1-\varepsilon)\frac{n}{r} \ge \frac{2|H|}{\varepsilon}$ for $i \in [r], t \le r \le T$. Consider a graph R on $[r]$ such that $ij \in E(R)$ if and only if (V_i, V_j) is ε-regular. Then

$$|E(R)| \ge \binom{r}{2} - \varepsilon r^2 \ge (1-3\varepsilon)\binom{r}{2} > (1 - \frac{1}{k-1})\binom{r}{2}.$$

By Turán's theorem, R contains a copy of K_k.

Now color the edges of K_k in the following way: color ij red if $d(V_i, V_j)$ has red density $\ge \frac{1}{2}$, and color blue otherwise. (Note that all pairs are ε-regular.) By the definition of $k = r(K_{\Delta+1})$, there exists a monochromatic copy of $K_{\Delta+1}$ in this K_k. Then, $V_1, V_2, \dots, V_{\Delta+1}$ are red(or blue) regular $(\Delta+1)$-tuple and the pair (V_i, V_j) is ε-regular for $i, j \in [\Delta + 1]$. Now we define graph G' as the red(or blue) graph on $V_1, \dots, V_{\Delta+1}$. Then G' is a $(\Delta+1)$-partite graph on $V_1, \dots, V_{\Delta+1}$ such that (V_i, V_j) is (ε, d)-regular with $d \ge \frac{1}{2}$ and $|H| \le \varepsilon|V_i|/2$. Apply the Graph embedding lemma to G' with $d = \frac{1}{2}$ and $m = |H|$, we can find a copy of H in G', which gives a monochromatic copy of H. Thus, there exists $c = c(\Delta)$ such that $r(H) \le c|H|$. □

Question: What about embedding large subgraphs or spanning subgraphs?

Definition 2.3. *Let G be a graph. A disjoint pair (A, B) of vertices is (ε, d)-super-regular if it's ε-regular, $d(A, B) \ge d$ and $d(a, B) \ge (d - \varepsilon)|B|, d(b, A) \ge (d-\varepsilon)|A|$ for all $a \in A, b \in B$.*

Lemma 2.4. *Let $2\varepsilon \le d \le 1$ and $n \ge 2/\varepsilon$. Let G be a graph. If (A, B) is (ε, d)-super-regular in G with $|A| = |B| = n$, then $G[A, B]$ contains a perfect matching.*

Theorem 2.5 (Blow-up lemma (Komlós-Sárközy-Szemerédi)). *Let $0 < \frac{1}{n} \ll \varepsilon \ll \frac{1}{r}, d_0, \frac{1}{\Delta} \le 1$. Suppose that H is an n-vertex graph satisfying vertex partition $X_1 \cup \cdots \cup X_r$ with $\Delta(H) \le \Delta$. Let G be a graph with partition $V_1 \cup \cdots \cup V_r$ such that $|V_i| = |X_i| = n$ and (V_i, V_j) is (ε, d')-super-regular for $d' \ge d$. Then we can embed H into G such that $\phi(X_i) = V_i$.*

Remark 2.6.

On the proof of the Blow-up lemma:

- *The proof of the embedding lemma (greedy embedding) can embed an* ε*-proportion of vertices.*
- *A careful randomized embedding (random greedy embedding) can embed an* $(1-\varepsilon)$*-proportion of vertices, succeeding with high probability.*
- *If we run the randomized embedding carefully, we can apply Hall-type result for the remaining vertices and obtain full embedding (Blow-up lemma).*

Exercise 2.7. *Suppose* $\varepsilon \ll d \leq 1$*. If* (A, B) *is* (ε, d)*-regular in* G*, then there exist* $A' \subseteq A$*,* $B' \subseteq B$ *such that* $|A'| \geq (1-\varepsilon)|A|$*,* $|B'| \geq (1-\varepsilon)|B|$ *and* (A', B') *is* $(2\varepsilon, d)$*-super-regular in* G*.*

Moving to hypergraph regularity

One of the main motivation of the hypergraph regularity is to understand/derive the hypergraph removal lemma. Recall that for $k \geq 2$, a *k-uniform hypergraph* H is a pair of (V, E), where V is a vertex set and E is a family of k-element subsets of V. For the convenience, we usually use *k-graph* to denote the k-uniform hypergraph.

Lemma 2.8 (Hypergraph removal lemma)**.** *For every* r*-graph* H *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that every* n*-vertex* r*-graph with* $< \delta n^{v(H)}$ *copies of* H *can be made* H*-free by removing* $< \varepsilon n^r$ *edges.*

Next we will give the following result, which is a corollary of the tetrahedron removal lemma.

Corollary 2.9. *If* G *is a* 3*-graph such that every edge is contained in a unique tetrahedron (i.e., a clique on four vertices), then* $e(G) = o(n^3)$*.*

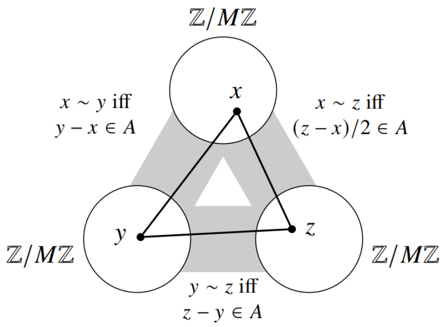
Now let's prove Roth's theorem and Szemerédi's Theorem for 4-AP, the application of triangle removal lemma. We write 3-AP for 3-term arithmetic progression. We say that A is *3-AP-free* if there are no $x, x+y, x+2y \in A$ with $y \neq 0$.

Theorem 2.10 (Roth's theorem)**.** *Let* $A \subseteq [N]$ *be 3-AP-free. Then* $|A| = o(N)$*.*

Proof. Embed $A \subseteq \mathbb{Z}/M\mathbb{Z}$ with $M = 2N+1$ (to avoid wraparounds). Since A is 3-AP-free in $\mathbb{Z}$, it is 3-AP-free in $\mathbb{Z}/M\mathbb{Z}$ as well.

Now, we construct a tripartite graph G whose parts X, Y, Z are all copies of $\mathbb{Z}/M\mathbb{Z}$. The edges of the graph are (since M is odd, we are allowed to divide by 2 in $\mathbb{Z}/M\mathbb{Z}$):

- $(x, y) \in X \times Y$ whenever $y - x \in A$;
- $(y, z) \in Y \times Z$ whenever $z - y \in A$;
- $(x, z) \in X \times Z$ whenever $(z - x)/2 \in A$.



In this graph, $(x,y,z) \in X \times Y \times Z$ is a triangle if and only if

$$y-x, \frac{z-x}{2}, z-y \in A.$$

The graph was designed so that the above three numbers form an arithmetic progression in the listed order. Since A is 3-AP-free, these three numbers must all be equal. So, every edge of G lies in a unique triangle, formed by setting the three numbers above to be equal.

The graph G has exactly $3M = 6N+3$ vertices and $3M|A|$ edges. As every edge lies in a unique triangle, G has exactly $3M|A|/3 = M|A| = o(M^3)$ triangles, and the triangle removal lemma says that G can be made triangle-free by removing $o(M^2)$ edges. However, as every edge of G is in a unique triangle, removing any edge destroys at most one triangle. That is, to make G triangle-free, one has to remove at least $M|A|$ edges. Therefore, we have $M|A| = o(M^2)$, yielding $|A| = o(M) = o(N)$ and we are done. □

In fact, Roth's theorem is the first case of a famous result known as Szemerédi's theorem.

Theorem 2.11 (Szemerédi's theorem)**.** *For every fixed $k \geq 3$, every k-AP-free subset of $[N]$ has size $o(N)$.*

Proof of Szemerédi's theorem for 4-AP. Let $A \subseteq [N]$ be 4-AP-free. Let $M = 6N+1$. Then A is also a 4-AP-free subset in $\mathbb{Z}/M\mathbb{Z}$. Build a 4-partite 3-graph G with parts W, X, Y, Z, all of which are copies of $[M]$. Define edges of G as follows, where w,x,y,z range over elements of W, X, Y, Z, respectively:

$$\begin{aligned} wxy \in E(G) &\iff 3w+2x+y \in A, \\ wxz \in E(G) &\iff 2w+x-z \in A, \\ wyz \in E(G) &\iff w-y-2z \in A, \\ xyz \in E(G) &\iff -x-2y-3z \in A. \end{aligned}$$

What is important here is that the ith expression does not contain the ith variable.

The vertices $xyzw$ form a tetrahedron if and only if

$$3w+2x+y,\ 2w+x-z,\ w-y-2z,\ -x-2y-3z \in A.$$

However, these values form a 4-AP with common difference $-x-y-z-w$. Since A is 4-AP-free, the only tetrahedra in A are trivial 4-APs (those with common difference zero). For each triple $(w,x,y) \in W \times X \times Y$, there is exactly one $z \in \mathbb{Z}/M\mathbb{Z}$ such that $x+y+z+w=0$. Thus, every edge of the hypergraph lies in exactly one tetrahedron.

By Corollary 2.9, the number of edges in the hypergraph is $o(M^3)$. On the other hand, the number of edges is exactly $4M^2|A|$ (for example, for every $a \in A$, there are exactly M^2 triples $(w,x,y) \in (\mathbb{Z}/M\mathbb{Z})^3$ with $3w+2x+y=a$). Therefore $|A| = o(M) = o(N)$. □