2025 BUPT Summer School - Course III

Regularity Methods and its Applications

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Lecture 1 The regularity method and the blowup lemma

As we all know, the regularity methods are some of the most powerful tools in combinatorics, which played a central role in graph theory, functional Analysis, ergodic theory and so on. Here, we firstly get to know one of the most classical applications of regularity lemma.

Lemma 1.1 (Triangle removal lemma)**.** *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that the following holds for large n. If G is an n-vertex graph with at most δn^3 triangles, then G can be made K_3-free by removing at most εn^2 edges.*

Next we can obtain the general result by extending triangle to any graph H.

Lemma 1.2 (Graph removal lemma)**.** *For any graph H and any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that any graph on n vertices which contains at most $\delta n^{v(H)}$ copies of H may be made H-free by removing at most εn^2 edges.*

Let $G = (V, E)$ be a graph. For disjoint sets $X, Y \subseteq V(G)$, the *edge-density* between X and Y is

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

Definition 1.3 (ε-regular)**.** *Given a graph G and some $\varepsilon > 0$, $V_1, V_2 \subseteq V(G)$, $V_1 \cap V_2 = \varnothing$. A pair (V_1, V_2) is called ε-regular if for any $A \subseteq V_1$ and $B \subseteq V_2$ with $|A| \geq \varepsilon|V_1|$, $|B| \geq \varepsilon|V_2|$, then $|d(A,B) - d(V_1,V_2)| < \varepsilon$.*

In particular, for convenience, we call a pair (V_1, V_2) (ε, d)-regular if it is ε-regular and $d(V_1, V_2) = d$.

Exercise 1.4. *Given $\varepsilon, c > 0$ and $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ with $|V_i'| \geq c|V_i|$. If (V_1, V_2) is ε-regular, then (V_1', V_2') is "also regular"* (that is, $\max\{2\varepsilon, \varepsilon/c\}$-regular).

Lemma 1.5 (K_3-Counting lemma)**.** *For every $\varepsilon > 0$, the following holds for large n. Suppose that there exist three disjoint vertex sets V_1, V_2, V_3 with $|V_i| \geq n$ such that for any $i, j \in [3]$, (V_i, V_j) is ε-regular and $d(V_i, V_j) \geq 2\varepsilon$. Then $G[V_1, V_2, V_3]$ contains at least $(1 - 2\varepsilon)(d_{12} - \varepsilon)(d_{13} - \varepsilon)(d_{23} - \varepsilon)|V_1||V_2||V_3|$ triangles.*

Proof. Take $V_1' \subseteq V_1$ such that $v \in V_1'$ if and only if $d(v, V_2) < (d_{12} - \varepsilon)|V_2|$ or $d(v, V_3) < (d_{13} - \varepsilon)|V_3|$. Then we claim that $|V_1'| \leq 2\varepsilon|V_1|$. Otherwise, if there exists a vertex set $V_{12} \subseteq V_1$ with $\varepsilon|V_1|$ vertices such that for any $v \in V_{12}$, $d(v, V_2) < (d_{12} - \varepsilon)|V_2|$, then we have $d(V_{12}, V_2) < \frac{(d_{12}-\varepsilon)|V_2||V_{12}|}{|V_{12}||V_2|} = d_{12} - \varepsilon$. But by the definition of ε-regular, we have $d(V_{12}, V_2) > d_{12} - \varepsilon$, a contradiction. Similarly, if there exists a vertex set $V_{13} \subseteq V_1$ with $\varepsilon|V_1|$ vertices such that for any $v \in V_{13}$, $d(v, V_3) < (d_{13} - \varepsilon)|V_3|$, then we have $d(V_{13}, V_3) < \frac{(d_{13}-\varepsilon)|V_3||V_{13}|}{|V_{13}||V_3|} = d_{13} - \varepsilon$. But by the definition of ε-regular, we have $d(V_{13}, V_3) > d_{13} - \varepsilon$, a contradiction. Thus, we derive that $|V_1'| \leq 2\varepsilon|V_1|$.

Now we consider the pair $(N(u) \cap V_2, N(u) \cap V_3)$. Note that $d(N(u) \cap V_2, N(u) \cap V_3) \in (d_{23} - \varepsilon, d_{23} + \varepsilon)$ because $N(u) \cap V_2 \subseteq V_2$ and $N(u) \cap V_3 \subseteq V_3$. Take any vertex $u \in V_1 \setminus V_1'$. Since $|N(u) \cap V_2| = d(u, V_2) \geq (d_{12} - \varepsilon)|V_2| \geq \varepsilon|V_2|$ and $|N(u) \cap V_3| = d(u, V_3) \geq (d_{13} - \varepsilon)|V_3| \geq \varepsilon|V_3|$, we have

$$\begin{aligned} e(N(u) \cap V_2, N(u) \cap V_3) &\geq (d_{23} - \varepsilon) \cdot |N(u) \cap V_2| \cdot |N(u) \cap V_3| \\ &\geq (d_{23} - \varepsilon)(d_{12} - \varepsilon)|V_2|(d_{13} - \varepsilon)|V_3|. \end{aligned}$$

Sum over all $u \in V_1 \setminus V_1'$, we get the number of K_3 in G is at least

$$(1 - 2\varepsilon)|V_1| \cdot (d_{23} - \varepsilon)(d_{12} - \varepsilon)|V_2|(d_{13} - \varepsilon)|V_3| = (1 - 2\varepsilon)(d_{12} - \varepsilon)(d_{13} - \varepsilon)(d_{23} - \varepsilon)|V_1||V_2||V_3|. \qquad \square$$

Remark 1.6.

- *Extend to K_r-counting in regular r-tuples by induction.*
- *Extend to F-counting in regular $\mathcal{X}(F)$-tuples, where $\mathcal{X}(F)$ is the chromatic number of G.*

Theorem 1.7 (Regularity lemma)**.** *For every $\varepsilon > 0$, $t \in \mathbb{N}$, there exist $N = N(\varepsilon, t)$ and $T = T(\varepsilon, t)$ such that the following holds for every $n \geq N$. Every n-vertex graph G admits an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_r$ with $t \leq r \leq T$,*

1. *$|V_i| = |V_j|$ for $1 \leq i, j \leq r$,*
2. *$|V_0| \leq \varepsilon n$,*
3. *(V_i, V_j) is ε-regular for all but at most εr^2 pairs with $i, j \in [r]$.*

Remark 1.8.

- *Only meaningful for dense graphs.*
- *$T = T(\varepsilon, t)$ is the upper bound of r, guaranteeing the "quality" of partition, but T is very large, which is $2^{2^{2^{\cdot^{\cdot^{2}}}}}$, where the height of the tower is a function of ε. Notice that the number of index is a function of ε and Gowers showed that this is unavoidable.*
- *Sometimes (or most of the time), you want to choose t large.*

Proof of triangle removal lemma. For every $\varepsilon > 0$, let ε be small and n be large. Suppose that G is a graph with less than δn^3 triangles.

Apply the regularity lemma with $t = 4/\varepsilon$ and $\delta = \frac{\varepsilon^3}{128T^3}$. Let $V_0 \cup V_1 \cup \cdots \cup V_r$ be the $\varepsilon/4$-regular partition with $t \le r \le T$.

Next, we will perform the following operation:

- remove all edges incident to V_0;
- remove all edges between irregular pairs;
- remove all edges inside each V_i with $i \in [r]$;
- remove all edges for (V_i, V_j) with $d(V_i, V_j) < \varepsilon/2$.

Thus, we removed at most

$$\begin{aligned}
&\frac{\varepsilon n}{4} \cdot (n-1) + \frac{\varepsilon r^2}{4} \cdot \left(\frac{n - |V_0|}{r}\right)^2 + r \cdot \binom{(n-|V_0|)/r}{2} + \binom{r}{2} \cdot \frac{\varepsilon}{2}\left(\frac{n-|V_0|}{r}\right)^2 \\
\le\ & \frac{\varepsilon n}{4} \cdot n + \frac{\varepsilon r^2}{4} \cdot \left(\frac{n}{r}\right)^2 + r \cdot \left(\frac{n}{r}\right)^2 + \binom{r}{2} \cdot \frac{\varepsilon}{2}\left(\frac{n}{r}\right)^2 \\
=\ & \frac{\varepsilon n^2}{4} + \frac{\varepsilon n^2}{4} + \frac{n^2}{2r} + \frac{\varepsilon n^2}{4} \\
=\ & \frac{3\varepsilon n^2}{4} + \frac{n^2}{2r} \le \varepsilon n^2
\end{aligned}$$

edges since $r \ge t = 4/\varepsilon$.

Let G' be the resulting graph. Now note that if $G' \supseteq K_3$, then there exist i, j, k such that this K_3 belongs to V_i, V_j, V_k and $(V_i, V_j), (V_i, V_k), (V_j, V_k)$ are all $\varepsilon/4$-regular with density $\ge \varepsilon/2$. Then the K_3-Counting lemma implies that $G'[V_i, V_j, V_k]$ has at least $(1-\varepsilon/2)(d_{ij}-\varepsilon/4)(d_{jk}-\varepsilon/4)(d_{ik}-\varepsilon/4)|V_i||V_j||V_k| \ge (1-\varepsilon/2)\cdot(\varepsilon/4)^3 \cdot \left(\frac{n-\varepsilon n/4}{r}\right)^3 > \frac{\varepsilon^3}{128T^3} n^3 = \delta n^3$ triangles, which contradicts with assumption. Thus, G' is K_3-free, that is, we obtain a K_3-free graph G' by removing at most εn^2 edges from G. $\square$

Remark 1.9. *Can we get better dependency between ε and δ? Improved bounds obtained by Fox (2011), by iterating Frieze-Kannan weak regularity.*

Other notable applications:

- $RT(K_4)$.
- If $\Delta(H) \le \Delta$, then $r(H) = O(|H|)$.
- Alon-Yuster theorem (by applying Blow-up lemma).

Application: Ramsey-Turán Theory

Question: If graph G is K_4-free and $\alpha(G) = o(n)$, then how many edges can G have? Szemerédi presented the following result.

Theorem 1.10 (Szemerédi)**.** *For any $\varepsilon > 0$, there exists $\alpha > 0$ such that the following holds for large n. If G is a K_4-free n-vertex graph and $\alpha(G) \le \alpha n$, then $e(G) \le (\frac{1}{8} + \varepsilon)n^2$.*

Proof. Let $\alpha = \frac{2\varepsilon^2}{25T}, t = \frac{5}{\varepsilon}$ and regularize graph G with $\varepsilon/5$. Then we get the following partition:

- $|V_0| \le \varepsilon n/5$,
- For all $1 \le i < j \le r$, $|V_i| = |V_j|$,
- (V_i, V_j) is $\varepsilon/5$-regular for all but at most $\varepsilon r^2/5$ pairs with $i, j \in [r]$.

Claim 1.11. *If (V_i, V_j) is ε-regular, then $d(V_i, V_j) < \frac{1}{2} + \frac{2\varepsilon}{5}$.*

Proof. Suppose that $d(V_i, V_j) \ge \frac{1}{2} + \frac{2\varepsilon}{5}$. Let $V_i' \subseteq V_i$ be the vertices that have degree $< (\frac{1}{2} + \frac{\varepsilon}{5})|V_j|$ to V_j. Then $|V_i'| \le \frac{\varepsilon}{5}|V_i|$. Thus, we have $|V_i \setminus V_i'| \ge (1 - \frac{\varepsilon}{5})|V_i| \ge (1 - \frac{\varepsilon}{5}) \cdot (1 - \frac{\varepsilon}{5})\frac{n}{r} \ge \frac{n}{2T} > \alpha n$. Since $\alpha(G) \le \alpha n$, we can pick an edge uv in $V_i \setminus V_i'$. Since $d(u, V_j), d(v, V_j) \ge (\frac{1}{2} + \frac{\varepsilon}{5})|V_j|$, we get

$$|N(u) \cap N(v) \cap V_j| \ge \frac{2\varepsilon}{5}|V_j| > \frac{2\varepsilon}{5} \cdot (1 - \frac{\varepsilon}{5})\frac{n}{r} > \frac{\varepsilon n}{5T} > \alpha n.$$

Then we can pick an edge in $N(u) \cap N(v)$, giving a $K_4 \subseteq G$, a contradiction. □

Next we define a d-Reduced graph R: Let R be a graph on $[r]$ such that $ij \in E(R)$ if and only if (V_i, V_j) is (ε, d')-regular with $d' \ge d$.

Let $d = 3\varepsilon/5$ and R be the d-reduced graph of the partition $(V_1, \ldots, V_r)$.

Claim 1.12. *R is K_3-free.*

Proof. Suppose not. Without loss of generality, there are three vertices 1,2,3 from V_1, V_2, V_3 forming a $K_3 \subseteq R$. Let $V_1' \subseteq V_1$ be vertex set such that for any vertex $v \in V_1'$, $d(v, V_2) < (d - \frac{\varepsilon}{5})|V_2|$ or $d(v, V_3) < (d - \frac{\varepsilon}{5})|V_3|$. Then $|V_1'| \le \frac{2\varepsilon}{5}|V_1|$.

Now we take a vertex $u \in V_1 \setminus V_1'$ and let $X = N(u) \cap V_2$, $Y = N(u) \cap V_3$. Note that $|X| \ge d(u, V_2) \ge (d - \frac{\varepsilon}{5})|V_2| \ge \frac{2\varepsilon}{5}|V_2|$. Let $X' \subseteq X$ be the vertex set such that for any vertex $w \in X'$, $d(w, Y) < (d - \varepsilon)|Y|$. By regularity, we get $|X'| \le \varepsilon|V_2|$, which implies that $|X \setminus X'| \ge |X| - \varepsilon|V_2| \ge \varepsilon|V_2|$.

Next we take any $v_2 \in X \setminus X'$, then

$$d(v_2, Y) \ge \left(d - \frac{\varepsilon}{5}\right)|Y| \ge \left(d - \frac{\varepsilon}{5}\right) \cdot \left(d - \frac{\varepsilon}{5}\right)|V_3| \ge \left(d - \frac{\varepsilon}{5}\right)^2 \left(\frac{n - \frac{\varepsilon n}{5}}{r}\right) \ge \frac{2\varepsilon^2}{25T}n > \alpha n.$$

Then we can pick an edge in $N(v_2, Y) = N(vv_2, V_3)$, giving a $K_4 \subseteq G$, a contradiction. □

Now we compute $e(G)$ by counting the following five parts:

- count all edges incident to V_0, which is at most $\varepsilon n^2/5$,
- count all edges between irregular pairs, which is at most $\frac{\varepsilon r^2}{5} \cdot (\frac{n}{r})^2 = \varepsilon n^2/5$,
- count all edges inside each V_i with $i \in [r]$, which is at most $r \cdot \binom{n/r}{2} \leq \frac{n^2}{2r} \leq \frac{n^2}{2t}$,
- count all edges for (V_i, V_j) with $d(V_i, V_j) < d$, which is at most $\binom{r}{2} \cdot d(\frac{n}{r})^2 \leq \frac{d}{2} n^2$,
- count all edges in R: Since R is K_3-free, by Mantel's theorem, $e(R) \leq \frac{r^2}{4}$ and each edge has density less than $\frac{1}{2} + \frac{2\varepsilon}{5}$. So the number of edges in R is at most $\frac{r^2}{4} \cdot (\frac{1}{2} + \frac{2\varepsilon}{5})(\frac{n}{r})^2 = (\frac{1}{8} + \frac{\varepsilon}{10})n^2$.

Adding all these up, we have

$$e(G) \leq 2\varepsilon n^2/5 + \frac{n^2}{2t} + \frac{dn^2}{2} + \left(\frac{1}{8} + \frac{\varepsilon}{10}\right)n^2 \leq \left(\frac{1}{8} + \varepsilon\right)n^2.$$

□

Remark 1.13.
- *Bollobás-Erdős found a graph saying that the bound* $1/8$ *is sharp.*
- *This theorem appeared before the Regularity lemma.*

Exercise 1.14. *Prove Erdős–Stone–Simonovits Theorem: Fix graph H with at least one edge. Then*

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$