Lecture 3: Quantum Multipartite Entanglement: k-Uniform States, Quantum Codes, and Graph States

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Contents

1 Basic Notation and Definitions

1.1 Quantum States and Reduced States

Definition 1.1 (q-level quantum system and pure state)**.** A *q-level quantum system* is described by the Hilbert space $\mathbb{C}^q$. Its computational orthonormal basis is denoted by

$$|0\rangle, |1\rangle, \ldots, |q-1\rangle.$$

Here a vector is written as a *ket* $|\psi\rangle$, and its conjugate transpose is written as the corresponding *bra* $\langle\psi|$. The inner product of two vectors is denoted by

$$\langle\phi|\psi\rangle := \langle\phi| \, |\psi\rangle.$$

In particular, $\langle i|j\rangle = \delta_{ij}$ for the computational basis.

A *pure state* of the system is a unit vector $|\psi\rangle \in \mathbb{C}^q$. With respect to the computational basis, it has a unique expansion

$$|\psi\rangle = \sum_{a=0}^{q-1} c_a |a\rangle, \quad \text{where} \quad c_a \in \mathbb{C}, \quad \text{and} \quad \sum_{a=0}^{q-1} |c_a|^2 = 1.$$

The coefficients c_a are called the *amplitudes* of $|\psi\rangle$.

Definition 1.2 (n-partite quantum system)**.** A system consisting of n q-level subsystems is described by the tensor-product Hilbert space

$$\mathcal{H} = (\mathbb{C}^q)^{\otimes n} = \underbrace{\mathbb{C}^q \otimes \cdots \otimes \mathbb{C}^q}_{n \text{ factors}}.$$

Its computational orthonormal basis consists of

$$|i_1 \cdots i_n\rangle := |i_1\rangle \otimes \cdots \otimes |i_n\rangle, \quad \text{where} \quad i_j \in \{0, \ldots, q-1\}.$$

A pure state of the whole system is therefore a unit vector $|\psi\rangle \in \mathcal{H}$, which can be written uniquely as

$$|\psi\rangle = \sum_{i_1,\ldots,i_n=0}^{q-1} c_{i_1,\ldots,i_n} |i_1 \cdots i_n\rangle, \quad \text{where} \quad \sum_{i_1,\ldots,i_n=0}^{q-1} |c_{i_1,\ldots,i_n}|^2 = 1.$$

Definition 1.3 (Product and entangled states)**.** A pure state $|\psi\rangle \in (\mathbb{C}^q)^{\otimes n}$ is a *product state* if it can be written as

$$|\psi\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle,$$

where each $|x_i\rangle \in \mathbb{C}^q$ is a unit vector. Otherwise, $|\psi\rangle$ is called an *entangled state*.

Definition 1.4 (Density matrix)**.** A quantum state of a quantum system is described by a positive semidefinite matrix ρ satisfying $\mathrm{Tr}(\rho) = 1$, where $\mathrm{Tr}(\rho)$ denotes the *trace* of ρ. The matrix ρ is called the *density matrix*. The state is *pure* if $\mathrm{rank}(\rho) = 1$, in which case $\rho = |\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle$. Otherwise, the state is called *mixed*.

Definition 1.5 (Reduced state)**.** Let ρ be the density matrix of an n-partite quantum system, and let $S \subseteq [n]$. The *reduced state* on the subsystems indexed by S is

$$\rho_S = \mathrm{Tr}_{S^c}(\rho),$$

where $S^c = [n] \setminus S$ and Tr_{S^c} denotes the partial trace over the subsystems in S^c.

For example, for an elementary tensor $A \otimes |u\rangle\langle v|$, the partial trace over subsystem B is

$$\mathrm{Tr}_B(A \otimes |u\rangle\langle v|) = A \, \mathrm{Tr}(|u\rangle\langle v|) = \langle v|u\rangle \, A.$$

Example 1.6 (Bell state)**.** Consider the two-qubit state

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Its density matrix is

$$\rho = \left|\Phi^+\right\rangle\left\langle\Phi^+\right| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|).$$

Tracing out the second qubit gives

$$\rho_{\{1\}} = \mathrm{Tr}_{\{2\}}(\rho) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I_2}{2}.$$

By symmetry, $\rho_{\{2\}} = I_2/2$ as well. Thus the two-qubit state is pure, while each individual qubit is in a mixed state.

The state I_d/d on a d-dimensional system is called the *maximally mixed state*. In particular, the maximally mixed state of r q-level systems is I_{q^r}/q^r. A useful measure of mixedness is the *purity* $\mathrm{Pur}(\rho) := \mathrm{Tr}(\rho^2)$. A pure state has purity 1, whereas the maximally mixed state on r q-level systems has $\mathrm{Pur}\left(\frac{I_{q^r}}{q^r}\right) = \frac{1}{q^r}$.

1.2 Absolutely Maximally Entangled States

Definition 1.7 (k-uniform state [2])**.** A pure state $|\psi\rangle \in (\mathbb{C}^q)^{\otimes n}$ is *k-uniform* if for every $S \subseteq [n]$ with $|S| = k$, $\rho_S = \frac{I_{q^k}}{q^k}$.

Proposition 1.8. *If a pure state is k-uniform, then it is also $(k-1)$-uniform.*

Proof. Let $T \subseteq [n]$ with $|T| = k-1$, and choose $i \notin T$. Since $\rho_{T\cup\{i\}} = I_{q^k}/q^k$, tracing out the ith subsystem gives $\rho_T = I_{q^{k-1}}/q^{k-1}$. □

Example 1.9 (GHZ state)**.** The n-qubit GHZ state is

$$|\mathrm{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle + |1^n\rangle).$$

Every one-qubit reduced state is $I_2/2$, so the state is 1-uniform. For $n \geq 3$, however,

$$\rho_{\{1,2\}} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \neq \frac{I_4}{4},$$

so it is not 2-uniform.

Theorem 1.10 (Schmidt decomposition)**.** *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces. Every unit vector $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as*

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |u_i\rangle_A \otimes |v_i\rangle_B\,,$$

where $\lambda_i > 0$, $\sum_i \lambda_i = 1$.

The families $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are orthonormal, and $r \leq \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}$. Moreover,

$$\rho_A = \sum_{i=1}^{r} \lambda_i\, |u_i\rangle\langle u_i|\,, \qquad \rho_B = \sum_{i=1}^{r} \lambda_i\, |v_i\rangle\langle v_i|\,.$$

Hence ρ_A and ρ_B have the same nonzero eigenvalues and the same rank.

Theorem 1.11 (Half-system bound)**.** *If a k-uniform state in $(\mathbb{C}^q)^{\otimes n}$ exists, then $k \leq \lfloor n/2 \rfloor$.*

Proof. Choose $S \subseteq [n]$ with $|S| = k$. Since $\rho_S = I_{q^k}/q^k$, we have $\operatorname{rank}(\rho_S) = q^k$. Across the bipartition $S \mid S^c$, Schmidt decomposition gives

$$q^k = \operatorname{rank}(\rho_S) = \operatorname{rank}(\rho_{S^c}) \leq q^{n-k}.$$

Hence $k \leq n - k$. □

Definition 1.12 (Absolutely maximally entangled state)**.** An n-partite pure state is *absolutely maximally entangled* if it is $\lfloor n/2 \rfloor$-uniform. We write $\mathrm{AME}(n, q)$ for such a state on n q-level systems.

For qubits, AME states exist precisely for $n \in \{2, 3, 5, 6\}$. The lecture also mentioned the existence of $\mathrm{AME}(8, 4)$ as an open problem.

2 Upper Bounds on k-Uniform States

The half-system bound gives $k \leq \lfloor n/2 \rfloor$. When the local dimension q is fixed, stronger upper bounds are known. We first state the main results discussed in the lecture, and then introduce the enumerator method used to obtain nonexistence results.

2.1 Known Upper Bounds

For qubits, Rains obtained the following asymptotic bound using quantum shadow enumerators.

Theorem 2.1 (Rains [5])**.** *If k-uniform states in $(\mathbb{C}^2)^{\otimes n}$ exist as $n \to \infty$, then*

$$\frac{k}{n} \leq \frac{1}{3} + o(1).$$

For qutrits, Shi, Ning, Zhao, and Zhang proved the corresponding bound.

Theorem 2.2 (Shi–Ning–Zhao–Zhang [7])**.** *If k-uniform states in $(\mathbb{C}^3)^{\otimes n}$ exist as $n \to \infty$, then*

$$\frac{k}{n} \leq \frac{3}{7} + o(1).$$

For states at the maximal possible uniformity $k = \lfloor n/2 \rfloor$, Scott obtained an explicit nonexistence bound.

Theorem 2.3 (Scott [2])**.** *An $\mathrm{AME}(n, q)$ state does not exist if*

$$n > \begin{cases} 2(q^2 - 1), & n \text{ even}, \\ 2q(q + 1) - 1, & n \text{ odd}. \end{cases}$$

More generally, write $k = \lfloor n/2 \rfloor - \ell$. The parameter ℓ measures the distance from the AME case and will be called the *defect*.

Theorem 2.4 (Zeng–Zhang [8])**.** *Let $q \geq 2$ and $\ell \geq 0$. If*

$$n > \begin{cases} 2(\ell + 1)q^2 + 2(\ell - 1), & n \text{ even}, \\ 2(\ell + 1)q^2 + 2(\ell + 1)q + (2\ell - 1), & n \text{ odd}, \end{cases}$$

then there is no $(\lfloor n/2 \rfloor - \ell)$-uniform state in $(\mathbb{C}^q)^{\otimes n}$.

For $\ell = 0$, this recovers Scott's bound. In particular, for fixed q and fixed defect, the number of parties cannot be arbitrarily large.

As a consequence, one obtains the following asymptotic upper bound for fixed local dimension.

Theorem 2.5 (Zeng–Zhang [8]). *Fix an integer $q \geq 2$. If there exists a k-uniform state in $(\mathbb{C}^q)^{\otimes n}$, then*

$$\frac{k}{n} \leq \begin{cases} \dfrac{q^2}{2(q^2+1)} + O\left(\dfrac{1}{n}\right), & n \text{ even}, \\ \dfrac{q^2+q}{2(q^2+q+1)} + O\left(\dfrac{1}{n}\right), & n \text{ odd}. \end{cases}$$

For even n, the leading constants are $2/5$ when $q = 2$ and $9/20$ when $q = 3$. These are weaker than the corresponding shadow bounds $1/3$ and $3/7$.

The rest of this section explains the algebraic framework behind such nonexistence results.

2.2 Pauli Expansion

To derive upper bounds, we need to convert the condition that many reduced states are maximally mixed into numerical constraints. For qubits, the basic tool is the Pauli basis

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These four matrices are orthogonal with respect to the Hilbert–Schmidt inner product $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$.

For n qubits, the tensor products $E = E_1 \otimes \cdots \otimes E_n$, where $E_i \in \{I, X, Y, Z\}$, form an orthogonal basis of all $2^n \times 2^n$ complex matrices. The *weight* of E is $\mathrm{wt}(E) = |\{i \in [n] : E_i \neq I\}|$. Thus $\mathrm{wt}(E)$ records the number of subsystems on which E acts nontrivially. Every n-qubit density matrix has the Pauli expansion

$$\rho = \frac{1}{2^n} \sum_E \mathrm{Tr}(E\rho)\, E.$$

For general local dimension q, choose an orthogonal unitary error basis $\mathcal{E}_q = \{e_0, e_1, \ldots, e_{q^2-1}\}$ of the $q \times q$ complex matrices such that $e_0 = I_q$,

$$\mathrm{Tr}(e_a^\dagger e_b) = q\delta_{ab},$$

and $\mathcal{E}_q$ is closed under taking adjoints up to phase. In particular, $\mathrm{Tr}(e_a) = 0$ for $a \neq 0$. Tensor products of these matrices form a basis $\mathcal{E}_n := \mathcal{E}_q^{\otimes n}$. For $E = E_1 \otimes \cdots \otimes E_n \in \mathcal{E}_n$, we use the same definition $\mathrm{wt}(E) = |\{i : E_i \neq I_q\}|$.

The following observation is the link between this expansion and k-uniformity.

Proposition 2.6 (Low-weight characterization). *Let $\rho = |\psi\rangle\langle\psi|$. Then $|\psi\rangle$ is k-uniform if and only if*

$$\mathrm{Tr}(E^\dagger \rho) = 0$$

for every nonidentity $E \in \mathcal{E}_n$ with $\mathrm{wt}(E) \leq k$.

Proof. Suppose first that $|\psi\rangle$ is k-uniform. Let $E \in \mathcal{E}_n$ be nonidentity with $\mathrm{wt}(E) \leq k$, and let $S = \mathrm{supp}(E)$. Since E is the identity outside S, $\mathrm{Tr}(E^\dagger\rho) = \mathrm{Tr}(E_S^\dagger \rho_S)$. Since $|S| \leq k$, the reduced state is maximally mixed, so

$$\mathrm{Tr}(E^\dagger\rho) = \frac{1}{q^{|S|}}\,\mathrm{Tr}(E_S^\dagger) = 0.$$

Conversely, suppose that all nonidentity coefficients of weight at most k vanish. For $S \subseteq [n]$ with $|S| \leq k$, expand ρ_S in the local basis. All nonidentity coefficients vanish, while the identity coefficient is $\mathrm{Tr}(\rho_S) = 1$. Hence $\rho_S = \frac{I_{q^{|S|}}}{q^{|S|}}$, so $|\psi\rangle$ is k-uniform. □

Thus k-uniformity forces all coefficients of weights $1, \ldots, k$ in the matrix expansion to vanish.

2.3 The Linear-programming Constraints

We now collect the coefficients of the same weight. For $\rho = |\psi\rangle\langle\psi|$, define

$$A_j := \sum_{\substack{E\in\mathcal{E}_n\\ \mathrm{wt}(E)=j}} \left|\mathrm{Tr}(E^\dagger\rho)\right|^2, \qquad 0\le j\le n.$$

Clearly $A_j \ge 0$ and $A_0 = 1$. By the preceding proposition, a k-uniform state satisfies $A_1 = \cdots = A_k = 0$.

A second enumerator is defined directly from the reduced states. Set

$$A'_j := \sum_{\substack{S\subseteq[n]\\ |S|=j}} \mathrm{Tr}(\rho_S^2), \qquad 0\le j\le n.$$

Since the global state is pure, the Schmidt decomposition across $S \mid S^c$ shows that ρ_S and ρ_{S^c} have the same nonzero eigenvalues. Hence $A'_j = A'_{n-j}$.

Moreover, a density matrix on a q^j-dimensional space has purity at least q^{-j}. Therefore $A'_j \ge q^{-j}\binom{n}{j}$. If $|\psi\rangle$ is k-uniform and $j\le k$, every j-party reduced state is maximally mixed, and hence equality holds: $A'_j = q^{-j}\binom{n}{j}$.

The two families of enumerators are related by the following identity.

Theorem 2.7 (Quantum MacWilliams identity [6])**.** *For every* $0\le i\le n$*,*

$$q^i A'_i = \sum_{j=0}^{i} A_j \binom{n-j}{i-j}.$$

Proof. For a fixed i-subset S, orthogonality of the local matrix basis gives

$$q^i\,\mathrm{Tr}(\rho_S^2) = \sum_{\mathrm{supp}(E)\subseteq S} \left|\mathrm{Tr}(E^\dagger\rho)\right|^2.$$

Summing over all $S\subseteq[n]$ with $|S| = i$, an operator of weight j is counted once for every i-subset containing its support. There are $\binom{n-j}{i-j}$ such subsets, which gives the identity. □

Combining the preceding relations, the existence of a k-uniform state implies the existence of nonnegative numbers $A_0,\dots,A_n$ and numbers $A'_0,\dots,A'_n$ satisfying

$$\begin{aligned}
&A_0 = 1, \qquad A_1 = \cdots = A_k = 0, \qquad A_i \ge 0, \\
&A'_i = q^{-i}\binom{n}{i} && (0\le i\le k),\\
&A'_i \ge q^{-i}\binom{n}{i} && (k< i\le \lfloor n/2\rfloor),\\
&A'_i = A'_{n-i} && (0\le i\le n),\\
&q^i A'_i = \sum_{j=0}^{i} A_j\binom{n-j}{i-j} && (0\le i\le n).
\end{aligned}$$

Thus nonexistence of a k-uniform state can be proved by showing that this linear system is infeasible. This is the linear-programming viewpoint behind the upper bounds above.

3 Constructions of k-Uniform States

3.1 Constructions from Orthogonal Arrays

Definition 3.1 (Orthogonal array)**.** Let Σ be an alphabet of size q. An *orthogonal array* $\mathrm{OA}(N, n, q, t)$ is an $N \times n$ array over Σ such that, in every choice of t columns, each ordered t-tuple in Σ^t occurs exactly N/q^t times.

Example 3.2. The array

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is an $\mathrm{OA}(4, 3, 2, 2)$.

Definition 3.3 (Irredundant orthogonal array)**.** An $\mathrm{OA}(N, n, q, t)$ is *irredundant* if, after deleting any t columns, the remaining $n - t$ entries distinguish all rows; equivalently, no two rows agree on all of the remaining $n - t$ columns.

Theorem 3.4 (Orthogonal-array construction [4])**.** *Let $\mathcal{A} = (a_{ij})$ be an irredundant $\mathrm{OA}(N, n, q, t)$, with rows $\mathbf{a}^{(i)} = (a_{i1}, \ldots, a_{in})$. Then*

$$|\psi_{\mathcal{A}}\rangle := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |a_{i1} \cdots a_{in}\rangle$$

is a t-uniform state.

Proof. Fix $S \subseteq [n]$ with $|S| = t$. In the reduced state ρ_S, irredundancy forces all off-diagonal terms to vanish after tracing out S^c. The orthogonal-array property says that every word in Σ^t occurs exactly N/q^t times on the coordinates in S. Hence

$$\rho_S = \frac{1}{q^t} \sum_{\mathbf{z} \in \Sigma^t} |\mathbf{z}\rangle \langle \mathbf{z}| = \frac{I_{q^t}}{q^t}.$$

□

Example 3.5 ($\mathrm{OA}(9, 4, 3, 2)$)**.** Over $\mathbb{F}_3$, let $\mathcal{A} = \{(i, j, i+j, i+2j) : i, j \in \mathbb{F}_3\}$. Every pair of columns contains each ordered pair exactly once, so this is an irredundant $\mathrm{OA}(9, 4, 3, 2)$. Therefore

$$|\psi\rangle = \frac{1}{3} \sum_{i,j \in \mathbb{F}_3} |i, j, i + j, i + 2j\rangle$$

is 2-uniform, and hence is an $\mathrm{AME}(4, 3)$ state.

3.2 Constructions from Graph States

We now describe a second construction of k-uniform states using graphs. Throughout this subsection, $G = (V, E)$ is a simple graph on n vertices. For $v \in V$, let $N_G(v)$ denote the neighborhood of v in G.

For one qubit, write $|+\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$. The controlled-$Z$ gate CZ acts on the computational basis by

$$CZ\,|ab\rangle = (-1)^{ab}\,|ab\rangle\,, \qquad a, b \in \{0, 1\}.$$

For an edge $\{u, v\} \in E$, we write CZ_{uv} for the controlled-Z gate acting on the qubits indexed by u and v.

Definition 3.6 (Graph state)**.** The *graph state* associated with G is

$$|G\rangle := \prod_{\{u,v\} \in E} CZ_{uv}\,|+\rangle^{\otimes n}.$$

To understand the uniformity of $|G\rangle$, we use its stabilizers. For a one-qubit Pauli matrix $M \in \{X, Z\}$ and $v \in V$, let M_v denote the operator that acts as M on the qubit indexed by v and as the identity on all other qubits. For each $v \in V$, define

$$K_v := X_v \prod_{u \in N_G(v)} Z_u.$$

These operators satisfy $K_v |G\rangle = |G\rangle$.

For $D \subseteq V$, define $K_D := \prod_{v \in D} K_v$. We also define the *odd neighborhood* of D by

$$\mathrm{Odd}_G(D) := \{u \in V : |N_G(u) \cap D| \text{ is odd}\}.$$

The purpose of $\mathrm{Odd}_G(D)$ is to describe exactly the support of the stabilizer K_D.

Lemma 3.7. *For every $D \subseteq V$, $\mathrm{supp}(K_D) = D \cup \mathrm{Odd}_G(D)$, and consequently $\mathrm{wt}(K_D) = |D \cup \mathrm{Odd}_G(D)|$.*

Proof. Each vertex in D contributes an X factor to K_D. On the other hand, a vertex u receives one Z_u factor from each neighbor of u that lies in D. Since $Z^2 = I$, a Z_u factor remains exactly when $|N_G(u) \cap D|$ is odd, that is, when $u \in \mathrm{Odd}_G(D)$.

If a vertex belongs to both D and $\mathrm{Odd}_G(D)$, then both an X and a Z act on that qubit, giving a Y factor up to phase. Hence the set of qubits on which K_D acts nontrivially is precisely $D \cup \mathrm{Odd}_G(D)$. □

This gives a purely graph-theoretic characterization of uniformity.

Theorem 3.8 (Graph-state criterion)**.** *The graph state $|G\rangle$ is k-uniform if and only if*

$$|D \cup \mathrm{Odd}_G(D)| \geq k + 1$$

for every nonempty $D \subseteq V$.

Proof. The operators K_D, $D \subseteq V$, form the stabilizer group of $|G\rangle$. A Pauli operator has nonzero expectation in $|G\rangle$ precisely when it is proportional to a stabilizer element; in that case its expectation has modulus one.

By the low-weight characterization of k-uniform states, $|G\rangle$ is k-uniform if and only if every nontrivial stabilizer has weight at least $k + 1$. By the preceding lemma, this is equivalent to $|D \cup \mathrm{Odd}_G(D)| \geq k + 1$ for every nonempty $D \subseteq V$. □

Thus the construction problem for k-uniform graph states has been reduced to finding graphs for which $|D \cup \mathrm{Odd}_G(D)|$ is large for every nonempty D.

Definition 3.9 (Local complementation)**.** For $v \in V$, the *local complementation* of G at v is obtained by replacing the subgraph induced by $N_G(v)$ with its complement, while leaving all other adjacencies unchanged. We write $G \equiv_{LC} G'$ if G' can be obtained from G by a sequence of local complementations.

Definition 3.10 (Local minimum degree)**.** The *local minimum degree* of G is

$$\delta_{\mathrm{loc}}(G) := \min\{\delta(G') : G' \equiv_{LC} G\},$$

where $\delta(G')$ denotes the minimum degree of G'.

The following characterization connects this graph parameter directly with the quantity appearing in the graph-state criterion:

$$\delta_{\mathrm{loc}}(G) = \min_{\emptyset \neq D \subseteq V} |D \cup \mathrm{Odd}_G(D)| - 1 \qquad [9].$$

Therefore,

$$|G\rangle \text{ is } k\text{-uniform} \iff \delta_{\mathrm{loc}}(G) \geq k.$$

Consequently, graphs with large local minimum degree give k-uniform states with large k. Probabilistic constructions yield graphs G_n satisfying $\delta_{\mathrm{loc}}(G_n) > 0.189n$ for all sufficiently large n [9]. Hence they give graph states whose uniformity is linear in the number of qubits.

The lecture also considered explicit constructions. One important example is the Paley graph.

Definition 3.11 (Paley graph)**.** Let $p \equiv 1 \pmod 4$ be a prime. The *Paley graph* Q_p has vertex set $\mathbb{F}_p$, where distinct $x, y \in \mathbb{F}_p$ are adjacent if and only if $x - y$ is a nonzero square in $\mathbb{F}_p$.

For the Paley graph, $\delta_{\mathrm{loc}}(Q_p) \geq \sqrt{p} - \frac{3}{2}$. Thus this explicit construction gives only a square-root lower bound, whereas the probabilistic construction gives a linear bound. This leads to the problem of finding an explicit infinite family of graphs G_n satisfying $\delta_{\mathrm{loc}}(G_n) \geq c|V(G_n)|$ for some absolute constant $c > 0$.

4 Quantum Error-Correcting Codes

We finally explain the connection between k-uniform states and quantum error-correcting codes. We use the local error basis $\mathcal{E}_n$ and the weight $\mathrm{wt}(E)$ introduced in the upper-bound section.

Definition 4.1 (q-ary quantum code)**.** A *q-ary quantum code* is a K-dimensional subspace $Q \subseteq (\mathbb{C}^q)^{\otimes n}$. Let P be the orthogonal projector onto Q. The code *detects* an error $E \in \mathcal{E}_n$ if $PEP = \lambda_E P$ for some scalar λ_E.

The equation $PEP = \lambda_E P$ means that the action of E cannot distinguish different states inside the code space.

Definition 4.2 (Distance and pure distance)**.** A quantum code has *distance at least* d if it detects every nonidentity error E with $\mathrm{wt}(E) < d$. It is *pure to distance* d if, in addition, $\lambda_E = 0$ for every nonidentity E with $\mathrm{wt}(E) < d$.

A q-ary quantum code of length n, dimension K, and distance d is denoted by $((n, K, d))_q$. When $K = q^r$, one also writes $[[n, r, d]]_q$.

For a one-dimensional code $Q = \mathbb{C}\,|\psi\rangle$, let $P = |\psi\rangle\langle\psi|$. Then $PEP = \langle\psi| E |\psi\rangle\, P$, so every error is automatically detected. In this case, the relevant condition is purity: low-weight nonidentity errors must have zero expectation.

Theorem 4.3 (k-uniform states and quantum codes [2])**.** *A pure state $|\psi\rangle \in (\mathbb{C}^q)^{\otimes n}$ is k-uniform if and only if the one-dimensional code $Q = \mathbb{C}\,|\psi\rangle$ is pure to distance $k+1$. Equivalently,*

$$|\psi\rangle \text{ is } k\text{-uniform} \iff Q \text{ is a pure } ((n, 1, k+1))_q \text{ code.}$$

Proof. Suppose first that $|\psi\rangle$ is k-uniform. Let $E \in \mathcal{E}_n$ be a nonidentity error with $\mathrm{wt}(E) \leq k$, and let $S = \mathrm{supp}(E)$. Then

$$\langle\psi| E |\psi\rangle = \mathrm{Tr}(E_S \rho_S) = \frac{1}{q^{|S|}} \mathrm{Tr}(E_S) = 0.$$

Hence the rank-one code is pure to distance $k+1$.

Conversely, suppose that $\langle\psi| E |\psi\rangle = 0$ for every nonidentity E of weight at most k. Let $S \subseteq [n]$ with $|S| \leq k$. Since $\mathcal{E}_q$ is closed under taking adjoints up to phase, the same vanishing condition holds for the coefficients $\mathrm{Tr}(E^\dagger \rho)$. Expanding the reduced density matrix ρ_S in the local matrix basis, all nonidentity coefficients vanish. Therefore $\rho_S = \frac{I_{q^{|S|}}}{q^{|S|}}$, so $|\psi\rangle$ is k-uniform. □

Thus the study of k-uniform states is equivalent, in the one-dimensional case, to the study of pure quantum codes with large distance. In particular, bounds and constructions for one side can be translated into corresponding results for the other.

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