Surface 2-cell embeddings of graphs, and group actions

Cai Heng Li

Southern University of Sciences and Technology

Riemann surfaces

- Orientable: a sphere with handles, uniquely determined by the number of handles: torus, double-torus, triple-torus, ...
- Non-orientable: a sphere with Mobius bands, uniquely determined by the number of mobius bands: projective plane, Klein bottle, ...

Maps

Definition:

- A 2-cell embedding of a graph in a surface partitions the surface into open Euclidean discs, called *faces*.
- The graph associated with the faces forms a *map*.

Example.

- The five regular polyhedrons are maps on the sphere.
- $\mathbf{K}_1^{(2)}$ embedded in a torus decomposes the torus into one single face.
- $\mathbf{K}_4$ embedded in a projective plane decomposes the plane into 3 faces; $\mathbf{K}_3^{(2)}$ embedded in a projective plane decomposes the plane into 4 faces, as the dual map.

Maps

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. Assume that Γ has a 2-cell embedding on a surface $\mathbb{S}$, which decomposes $\mathbb{S}$ into a face set F. Denote the map by

$$\mathcal{M} = (V, E, F),$$

in this case, $\mathbb{S}$ is the supporting surface, and Γ is the underlying graph.

Definition. The dual map of a map $\mathcal{M} = (V, E, F)$ is the map $\mathcal{M}^* = (F, E, V)$.

An arc is an incident pair (α, e) be a vertex α and an edge e.

Automorphisms of a map

Let $\mathcal{M} = (V, E, F)$ be a map.

Definition. A permutation of the set $V \cup E \cup F$ is called an automorphism of $\mathcal{M}$ if it preserves V, E, F and the incidence relation between vertices, edges and faces.

All automorphisms of $\mathcal{M}$ form the automorphism group $\mathrm{Aut}\mathcal{M}$.

Definition. If $G \leq \mathrm{Aut}\mathcal{M}$ is transitive on E, then $\mathcal{M}$ is called edge-transitive; similarly, we cen define vertex-transitive, and face-transitive.

Automorphisms

Let $\mathcal{M}$ be a map, and let $G \leq \mathrm{Aut}\mathcal{M}$.
Let $e = [\alpha, e, \beta]$ be an edge, and f, f' the faces incident with e.
Since an automorphism of $\mathcal{M}$ fixing α is a rotation or a reflection,

- G_α is cyclic or dihedral, and is faithful on $E(\alpha)$.

The four flags incident with the edge e are

$$(\alpha, e, f),\ (\alpha, e, f'),\ (\beta, e, f),\ (\beta, e, f').$$

Since G_e permutes the four flags and preserves their incidence, it yields

- $G_e \leq \mathrm{D}_4$, and is faithful on the 4 flags incident with e.

A map with single vertex or a single face

Let $\mathcal{M} = (V, E, F)$ be an edge-transitive map, with the underlying graph $\varGamma = (V, E)$. Then either

1. $\mathcal{M}$ has a single vertex with $\varGamma = \mathbf{K}_1^{(\lambda)}$, or a single face with $\varGamma^* = \mathbf{K}_1^{(\lambda)}$, or
2. the two sides of each edge are different faces, and $|V|, |F| \geq 2$.

In the general case (2), a flag is an incidence triple (α, e, f) of a vertex α, an edge e and a face f.

An automorphism fixing a flag must fix everything of the map, and is the identity. So a flag-transitive map is flag-regular, so called a regular map.

Regular maps

Regular maps

Let $\mathcal{M} = (V, E, F)$ be a regular map with $|V|, |F| \geq 2$, and $G = \mathrm{Aut}\mathcal{M}$.

Fixing a flag (α, e, f), there exist involutions x, y, z with $xz = zx$ such that

$$G = \langle x, y, z \rangle,$$
$$G_\alpha = \langle x, y \rangle,\ G_f = \langle y, z \rangle \text{ and } G_e = \langle x, z \rangle \cong \mathrm{D}_4.$$

Combinatorial incidence configurations

Definition. Given a group $G = \langle x, y, z \rangle$ where x, y, z are involutions with $xz = zx$, define an incidence configuration (V, E, F), where

$$V = [G : \langle x, y \rangle],\ E = [G : \langle x, z \rangle],\ F = [G : \langle y, z \rangle]$$

such that the incidence relation between $\alpha \in V$, $e \in E$ and $f \in F$ are defined by non-empty intersection of the corresponding cosets.

Then G acts transitively on V, E and F by right multiplication.

Incidence graphs

Claim: $\Gamma = (V, E)$ is a graph:

since the 'edge' $\langle x, z\rangle g \in E$ is incident with the two 'vertices' $\langle x, y\rangle g, \langle x, y\rangle zg \in V$, g.r.t. an edge

$$\langle x, z\rangle g = [\langle x, y\rangle g,\ \langle x, z\rangle g,\ \langle x, y\rangle zg]$$

Combinatorial maps

(1). The edges incident with the 'cycle' $\langle y, z\rangle$ are

$$\langle x, z\rangle\rho^i, \text{ where } \rho = yz, \text{ and } 0 \leq i \leq |\rho| - 1,$$

forming a cycle $C = \langle x, z\rangle^{\langle\rho\rangle}$: spinning the edge $e = \langle x, z\rangle$ by $\langle\rho\rangle$:

$$\begin{array}{ccccccc} \ldots, & \langle x, z\rangle\rho^{-1}, & \langle x, y\rangle, & \langle x, z\rangle, & \langle x, y\rangle\rho, & \langle x, z\rangle\rho, & \ldots, \\ \ldots, & e^{\rho^{-1}}, & \alpha, & e, & \alpha^{\rho}, & e^{\rho}, & \ldots, \end{array}$$

(2). The collection $\mathcal{C} = \{C^g \mid g \in G\}$ is a cycle-double-cover of Γ, since each edge $\langle x, z\rangle g$ is incident with $\langle y, z\rangle g$ and $\langle y, z\rangle xg$.

Cycle-double-covers of a graph

Given a graph $\Gamma = (V, E)$, a collection $\mathcal{C}$ of cycles of Γ is called a *cycle-double-cover* (a CDC) if

- each edge of Γ appears in exactly two cycles in $\mathcal{C}$, or equivalently,
- each edge of Γ is covered two times by the cycles in $\mathcal{C}$.

Observation: In the general case, the boundary cycles of the faces of a map form a cycle-double-cover of the underlying graph.

However, a cycle-double-cover does not necessarily form the boundary cycles of a map.

Combinatorial maps

Definition: Given a group $G = \langle x, y, z \rangle$,

- the coset graph $\mathrm{Cos}(G, \langle x, y \rangle, \langle x, z \rangle) = (V, E)$, and
- the set $\mathcal{C}$ of cycles

define an incidence configuration $(V, E, \mathcal{C})$ with the natural incidence relations, called a combinatorial map, and denoted by

$$\mathcal{M}(G, x, y, z) = (V, E, \mathcal{C})$$

A topological space

Lemma. Given $\mathcal{M}(G, x, y, z) = (V, E, \mathcal{C})$, identifying each cycle $\langle y, z\rangle g \in \mathcal{C}$ with an open Euclidean disc defines

- a closed Riemann surface $\mathbb{S}$, and
- a 2-cell embedding of $\Gamma = (V, E)$ in $\mathbb{S}$.

Let F denote the set of discs corresponding to the cycles in $\mathcal{C}$. Then $\mathcal{M} = (V, E, F)$ is a regular map, with $\mathrm{Aut}\mathcal{M} = G$.

Riemann surfaces

- A *Riemann surface* is a topological space of which sufficient small neighborhoods of each point are Euclidean discs.
- A big theorem of Topology shows that a surface is
  - either orientable, or non-orientable, and
  - uniquely determined by the genus.

Regular coset maps

Let $G = \langle x, y, z \rangle$, and let $\mathcal{M} = (V, E, F)$ be as constructed above.

Lemma. The triple (V, E, F) forms a regular map with vertex set V, edge set E and face set F.

Theorem. Each regular map is isomorphic to a regular coset map.

A class of edge-regular maps

2-Generated groups

$$G = \langle b, w \rangle$$

Example. Each finite simple group is generated by two elements.

Definition (coset graphs). Given $G = \langle b, w\rangle$, define a graph $\Gamma = (V, E)$, where

$$
\begin{aligned}
V &= [G : \langle b\rangle] \cup [G : \langle w\rangle], \\
E &= G
\end{aligned}
$$

such that $\langle b\rangle x \sim g \iff g \in \langle b\rangle x$, and $\langle w\rangle y \sim g \iff g \in \langle w\rangle y$.

The graph is a *coset graph*, denoted by $\mathrm{Cos}(G, \langle b\rangle, \langle w\rangle)$.

Cycles

Given a group $G = \langle b, w \rangle$, define a cycle $C(b, w)$ of $\mathrm{Cos}(G, \langle b \rangle, \langle w \rangle)$:

$$\langle w \rangle, 1, \langle b \rangle, b^{-1}, \ldots, \langle w \rangle (bw)^{-i}, (bw)^{-i}, \langle b \rangle (bw)^{-i}, b^{-1}(bw)^{-i}, \ldots, \langle b \rangle w, w, \langle w \rangle,$$

obtained by spinning the 2-arc $(\langle w \rangle, 1, \langle b \rangle, b^{-1})$ by the cyclic group $\langle bw \rangle$. Since the right multiplication of the group G induces a group of automorphisms of Γ, for any element $g \in G$, the image $C(b, w)g$ of $C(b, w)$ under g is also a directed cycle of Γ, which is

$$\ldots, \langle w \rangle (bw)^{-i} g, (bw)^{-i} g, \langle b \rangle (bw)^{-i} g, b^{-1}(bw)^{-i} g, \ldots.$$

Cycles

Let

$$\mathcal{C} = \{C(b,w)g \mid g \in G\},$$

a set of cycles of $\Gamma = \mathrm{Cos}(G, \langle b \rangle, \langle w \rangle)$.

Lemma. The cycles in $\mathcal{C}$ form a cycle-double-cover of Γ.

Arguing as for regular maps yields that the graph Γ associated with the cycle set $\mathcal{C}$ gives rise to an edge-regular map.

Three types of geometrices

Three types of geomtrices

The values of

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell}$$

divide triangle groups (and maps, surfaces) into three classes, i.e,

$\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell}$	>1	$=1$	<1
type	spherical	Euclidean	hyperbolic

Without loss of generality, assume $1 \leqslant m \leqslant n \leqslant \ell$, and $\ell \geqslant 2$.

Spherical

For $\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell}>1$, as $\ell \geqslant 2$, we have $\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell} \leqslant 2$. It yields $\chi = 2$, and $g = 0$, a sphere on which the regular maps are

(m,n,ℓ)	$(1,n,n)$	$(2,2,\ell)$	$(2,3,3)$	$(2,3,4)$	$(2,3,5)$
$T_{m,n,\ell}$	C_n	$D_{2\ell}$	A_4	S_4	A_5
$\mathcal{M}(G,a,b)$	$\mathbf{K}_{1,n}$	C_n	Tetrahedron (subdivison)	Cube Octehedron	Dodecahedron Icasohedron

A sphere can be tessellated using a triangle with angles $\frac{\pi}{m}$, $\frac{\pi}{n}$ and $\frac{\pi}{\ell}$, the sum $\frac{\pi}{m}+\frac{\pi}{n}+\frac{\pi}{\ell}>\pi$.

Euclidean

If $\frac{1}{m} + \frac{1}{n} + \frac{1}{\ell} = 1$, then $\{m, n, \ell\} = \{2, 3, 6\}$, $\{2, 4, 4\}$ or $\{3, 3, 3\}$, and

$$\begin{aligned} \chi(\mathbb{S}) &= 2 - 2\mathrm{g}(\mathbb{S}) = |G| \left(\frac{1}{m} + \frac{1}{n} + \frac{1}{\ell} - 1 \right) = 0, \\ \mathrm{g}(\mathbb{S}) &= 1, \quad \text{torus}. \end{aligned}$$

Moreover,

- $T_{m,n,\ell}$ is infinite and solvable (abelian-by-cyclic);
- (Magnus 1974) $T_{m,n,\ell} = \mathbb{Z}^2{:}\mathbb{Z}_\ell$, where $\ell = 3$, 4 or 6;
- the Euclidean plane can be tessellated using a triangle with angles $\frac{\pi}{m}$, $\frac{\pi}{n}$ and $\frac{\pi}{\ell}$, the sum $\frac{\pi}{m} + \frac{\pi}{n} + \frac{\pi}{\ell} = \pi$.

Hyperbolic

For the case $\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell}<1$, the pair (χ, g) satisfies

$$\begin{aligned} \chi(\mathbb{S}) &= 2-2\mathrm{g}(\mathbb{S}) = |G|\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell}-1\right) < 0, \\ \mathrm{g}(\mathbb{S}) &\geqslant 2. \end{aligned}$$

Furthermore, by Magnus (1974),

- $T_{m,n,\ell}$ is infinite and non-solvable;
- the hyperbolic plane can be tessellated using a triangle with a hyperbolic triangle with angles $\frac{\pi}{m}$, $\frac{\pi}{n}$ and $\frac{\pi}{\ell}$, where $\frac{\pi}{m}+\frac{\pi}{n}+\frac{\pi}{\ell}<\pi$.

Question: What can we say about hyperbolic triangle groups $T_{m,n,\ell}$?

Regular complete maps

Let $\mathcal{M} = (V, E, F)$ be a regular map, and let $G = \mathrm{Aut}\mathcal{M}$. Assume that the underlying graph $\Gamma = (V, E) = K_n$. Then G is 2-transitive on V, and G_α is a dihedral subgroup.
An inspection of 2-transitive permutation groups leads to

$$G/K = \mathrm{S}_3,\ \mathrm{S}_4,\ \mathrm{A}_5,$$

with stabiliser isomorphic to D_2, D_6 or D_{10}, respectively.

A classification

First, assume that $\Gamma = \mathbf{K}_2^{(\lambda)}$. Then

$$G = G_\alpha G_e = \langle x, y \rangle \langle z \rangle,$$

$G/K \cong \mathrm{S}_2$. Thus $K = \mathbb{Z}_\lambda$ or D_λ, and $G = \mathbb{Z}_\lambda.\mathbb{Z}_2$ or $\mathrm{D}_\lambda.\mathbb{Z}_2$, respectively. Since G is generated by 3 involutions, it follows that $G = \mathrm{D}_{2\lambda}$ or $\mathrm{S}_2 \times \mathrm{D}_\lambda$. Now, assume that $\Gamma = \mathbf{K}_3^{(\lambda)}$. Then $G/K \cong \mathrm{S}_3$, and $K = \mathbb{Z}_\lambda$ or D_λ. Thus $G = \mathbb{Z}_\lambda.\mathrm{S}_3 = (\mathbb{Z}_\lambda \times \mathbb{Z}_3).\mathbb{Z}_2$ or $\mathbb{Z}_{3\lambda}.\mathbb{Z}_2$.

Theorem. Let $\Gamma = (V, E)$ be a complete mutli-graph $\mathbf{K}_n^{(\lambda)}$ of order $n \geq 4$. Let $\mathcal{M}$ be a flag-regular embedding of Γ, and $G = \mathrm{Aut}\mathcal{M}$. Then either

1. $\Gamma = \mathbf{K}_4^{(\lambda)}$, and $G = 2^2{:}\mathrm{D}_{6\lambda}$, or $(\mathbb{Z}_\lambda \circ 8).\mathrm{D}_6$ with λ divisible by 4, or
2. $\Gamma = \mathbf{K}_6$ or $\mathbf{K}_6^{(2)}$, and $G = \mathrm{A}_5$ or $2 \times \mathrm{A}_5$, respectively.

Euler formula

The Euler characteristic and the genus of a surface are related to each other:

$$\chi(\mathbb{S}) = \begin{cases} 2 - 2\mathrm{g}(\mathbb{S}), & \text{for orientable surfaces,} \\ 2 - \mathrm{g}(\mathbb{S}), & \text{for non-orientable surfaces.} \end{cases}$$

In particular, if $\chi(\mathcal{M})$ is odd then $\mathcal{M}$ is non-orientable.

Coverings of groups, maps and surfaces

Triangle groups

$$T_{m,n,\ell} = \langle b, w \mid b^m = w^n = (bw)^\ell \rangle, \quad \text{where } \frac{1}{m} + \frac{1}{n} + \frac{1}{\ell} < 1.$$

Problem: Characterize finite quotients of a triangle group. For instance, given (m, n, ℓ), which finite simple groups are quotients of $T_{m,n,\ell}$?

Groups, graphs, maps, and surfaces

- finite (m, n, ℓ)-group $G = \langle b, w \rangle$, where $(m, n, \ell) = (|b|, |w|, |bw|)$;
- coset graph $\mathrm{Cos}(G, \langle b \rangle, \langle w \rangle)$;
- a map $\mathcal{M}(G, b, w)$ of type (m, n, ℓ);
- a Riemann surface with Euler characteristic

$$\begin{aligned} \chi = |V| - |E| + |F| &= |G|(\tfrac{1}{|b|} + \tfrac{1}{|w|} - 1 + \tfrac{1}{|bw|} \\ &= |G|(\tfrac{1}{m} + \tfrac{1}{n} + \tfrac{1}{\ell} - 1). \end{aligned}$$

Smooth covers

For a member $G \in \mathcal{T}_{m,n,\ell}$, if a quotient $\overline{G} = \langle \overline{b}, \overline{w} \rangle \in \mathcal{T}_{m,n,\ell}$, then

$$\begin{aligned} G &= \langle b, w \rangle, \text{ where } |b| = m,\ |w| = n, \text{ and } |bw| = \ell, \\ \overline{G} &= \langle \overline{b}, \overline{w} \rangle, \text{ where } |\overline{b}| = m,\ |\overline{w}| = n, \text{ and } |\overline{b}\overline{w}| = \ell. \end{aligned}$$

In this case,

- $\overline{G}$ is called a smooth quotient of G, and
- G is called a smooth cover of $\overline{G}$.

Smooth covers of maps

With (G, b, w) and $(\overline{G}, \overline{b}, \overline{w})$ given above, let

$$\begin{aligned} \mathcal{M} &= \mathcal{M}(G, b, w), \text{ and} \\ \mathcal{M}_N &= \mathcal{M}(\overline{G}, \overline{b}, \overline{w}). \end{aligned}$$

If (G, b, w) is a smooth cover of $(\overline{G}, \overline{b}, \overline{w})$, then $\mathcal{M}$ and $\mathcal{M}_N$ have the bi-valency $(|\overline{b}|, |\overline{w}|) = (|b|, |w|)$ and face length $|\overline{b}\overline{w}| = |bw|$; in this case,

Definition. $\mathcal{M}(G, b, w)$ is called a smooth cover of $\mathcal{M}(\overline{G}, \overline{b}, \overline{w})$

Quotients and coverings of surfaces

For orientable (closed) surfaces $\mathbb{S}$ and $\mathbb{S}'$, if there is a continuous function

$$\varphi : \begin{array}{ccc} \mathbb{S} & \to & \mathbb{S}' \\ \{n \text{ points}\} & \mapsto & \{1 \text{ point}\}. \end{array}$$

such that each point of $\mathbb{S}'$ has precisely n preimages in $\mathbb{S}$, then $\mathbb{S}$ is said to be a *n-sheeted smooth covering* of $\mathbb{S}'$.

Lemma. The following are equivalent:

1. the group (G, b, w) is a smooth cover of the quotient group $(\overline{G}, \overline{b}, \overline{w})$;
2. the map $\mathcal{D}(G, b, w)$ is a smooth cover of $\mathcal{D}(\overline{G}, \overline{b}, \overline{w})$;
3. the surface $\mathbb{S}$ is a smooth cover of the quotient surface $\mathbb{S}'$.

Euler characteristic

For a map $\mathcal{D} = (V, E, F)$, the Euler characteristic

$$\chi(\mathcal{D}) = |V| - |E| + |F|,$$

and the genus is given by $\chi(\mathcal{D}) = 2 - 2\mathrm{g}(\mathcal{D})$.

Let $\mathcal{D} = \mathcal{D}(G, b, w)$ be a smooth cover of $\mathcal{D}_N = \mathcal{D}(\overline{G}, \overline{b}, \overline{w})$. Then

$$\begin{aligned}
\chi(\mathcal{D}) &= |V| - |E| + |F| \\
&= \frac{|G|}{|b|} + \frac{|G|}{|w|} - \frac{|G|}{1} + \frac{|G|}{|bw|} \\
&= |G|(\frac{1}{|b|} + \frac{1}{|w|} - 1 + \frac{1}{|bw|}) \\
&= |N||\overline{G}|(\frac{1}{|\overline{b}|} + \frac{1}{|\overline{w}|} - 1 + \frac{1}{|\overline{b}\overline{w}|}) \\
&= |N|.\chi(\mathcal{D}_N).
\end{aligned}$$

Example

Let $G = \langle b, w \rangle$, a nonabelian p-group of order p^3 and exponent p. Then $|b| = |w| = |bw| = p$, and $N := \mathbf{Z}(G) = G' \cong \mathbb{Z}_p$. Let $\mathcal{D} = \mathcal{D}(G, b, w)$.

- $\mathcal{D}$ has $p^2 + p^2$ vertices, and p^2 faces, and $\chi(\mathcal{D}) = p^2(3 - p)$.
- $\overline{G} = G/N = \langle \overline{b}, \overline{w} \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, and $\mathcal{D}_N = \mathcal{D}(\overline{G}, \overline{b}, \overline{w})$ has $2p$ vertices, p^2 edges, p faces, and $\chi(\mathcal{D}_N) = p(3 - p)$.
- $\mathbb{S}_{p^2(3-p)}$ is a p-smooth cover of $\mathbb{S}_{p(3-p)}$.

Face-quasiprimitive regular dessins

A permutation group $G \leq \mathrm{Sym}(\Omega)$ is said to be *quasiprimitive* if any non-trivial normal subgroup is transitive.

Theorem (Chen, Fan, Li, Zhu 2026) Let $\mathcal{D}$ be a regular dessin, and let $G = \mathrm{Aut}\mathcal{D}$ be face-quasiprimitive and $N = \mathrm{soc}(G)$. If $\mathcal{D}$ is a smooth cover of $\mathcal{D}_N$, then G/N is cyclic, and $G = N{:}\mathbb{Z}_\ell$ is of type HA, TW or AS.

Conversely, in each of the following cases, G is a smooth cover of $\mathbb{Z}_\ell$:

1. $G = N{:}\mathbb{Z}_\ell$ is of type HA, with $\ell \geq 3$, or
2. $G = T \wr \mathbb{Z}_\ell = T^\ell{:}\mathbb{Z}_\ell$ is of type TW, where $\ell \geq 5$, or
3. $G = \Sigma\mathrm{L}(2, 2^\ell) = \mathrm{SL}(2, 2^\ell){:}\mathbb{Z}_\ell$, where $\ell \geq 5$ is a prime.

Schur coverings

Let $G = \mathrm{SL}(2,p)$, and $\overline{G} = G/N = \mathrm{PSL}(2,p)$, where $N = \mathbf{Z}(G) = \mathbb{Z}_2$.

Then $G = \langle b, w\rangle$, where

$$b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ w = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then $|b| = |w| = p$, and $\mathcal{D} = \mathcal{D}(G, b, w)$ is of valency p. Further,

> $G = \langle b, w\rangle$ is a smooth cover of $\overline{G} = \langle \overline{b}, \overline{w}\rangle \iff |bw|$ is odd;
>
> namely, $(|\overline{b}|, |\overline{w}|, |\overline{bw}|) = (|b|, |w|, |bw|) \iff |bw|$ is odd.

However, it is non-trivial to determine whether $|bw|$ is odd or not.

Fibonacci matrix

Notice that $bw = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and $(bw)^k$ is a Fibonacci matrix:

$$(bw)^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^k = \begin{bmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1} \end{bmatrix},$$

where $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-2} + F_{i-1}$, that is, F_i is a Fibonacci number. Thus the order $|ab|$ is determined by the Fibonacci numbers.

Fibonacci sequence

Depends on properties of Fibonacci sequence given in D.Wall (1960), J.Vinson (1963), and A.Dishong and M.Renault (2018).

Lemma. Let $p \geqslant 5$ be a prime. Then the following hold:

(i) if $p = 5$, then bw is of order 10;

(ii) if $p \equiv \pm 1 \pmod 5$, then $|ab|$ divides $\frac{p-1}{2}$; in particular, bw is of odd order if $p \equiv 3 \pmod 4$;

(iii) if $p \equiv \pm 2 \pmod 5$, then $|bw|$ divides $p+1$ but $|bw| \nmid \frac{p+1}{2}$, so that bw is of even order.

Fibonacci covers

Proposition. Let $G = \mathrm{SL}_2(p)$, and $\overline{G} = \mathrm{PSL}_2(p)$. Let

$$\mathcal{D} = \mathcal{D}(G, b, w), \text{ and } \overline{\mathcal{D}} = \mathcal{D}(\overline{G}, \overline{b}, \overline{w}).$$

Then

(i) $\mathcal{D}$ is a smooth cover of $\overline{\mathcal{D}}$ when $p \equiv 11$ or $19 \pmod{20}$;
infinitely many possibilities.

(ii) $\mathcal{D}$ is not a smooth cover of $\overline{\mathcal{D}}$ when $p \equiv \pm 2 \pmod{5}$;
infinitely many possibilities.

Example. Notice that the 5-th power $(bw)^5 = \begin{bmatrix} 34 & 55 \\ 55 & 89 \end{bmatrix}$.

(a) $(bw)^5 = I$ for $p = 11$, so $\mathcal{D}(G, b, w) \to \mathcal{D}(\overline{G}, \overline{b}, \overline{w})$ is smooth, $\mathbb{S}_{409}$ is a smooth double covering of $\mathbb{S}_{205}$.

(b) $(bw)^5 = -I$ for $p = 5$, so $\mathcal{D}(G, b, w) \to \mathcal{D}(\overline{G}, \overline{b}, \overline{w})$ is not smooth: $\mathbb{S}_{31} \to \mathbb{S}_{13}$ with 12 branched points.

- A surface $\mathbb{S}_{409}$ is a smooth double covering of $\mathbb{S}_{205}$.

$$\begin{array}{rcl} \mathbb{S}_{409} & \to & \mathbb{S}_{205} \\ \{2 \text{ points}\} & \mapsto & 1 \text{ point} \end{array}$$

This is realized by the quotient $\mathrm{SL}_2(11) \to \mathrm{PSL}_2(11)$.