Graphs Embeddings and Group Actions

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1 Surface 2-cell embeddings of graphs

A closed surface can be divided into two classes according to its orientability: orientable surfaces and non-orientable surfaces.

A surface is *orientable*if it can be viewed as a sphere with some handles attached. For example, according to the number of handles, we have

- A *sphere* has no handle.
- A *torus* is a sphere with one handle.
- A *double-torus* is a sphere with two handles.
- A *triple-torus* is a sphere with three handles.

Here, attaching a handle may be understood as removing two disjoint open discs from the sphere and joining the two resulting boundary circles by a tube. Repeating this construction gives the other orientable surfaces. Thus, an orientable surface is uniquely determined by the number of handles.

A surface is *non-orientable* if it can be viewed as a sphere with some Möbius bonds attached. For example, according to the number of Möbius bonds, we have

- A *projective plane* is a sphere with one Möbius bond.
- A *Klein bottle* is a sphere with two Möbius bonds.

More precisely, attaching a Möbius bond may be understood as removing an open disc from the sphere and gluing the boundry of a Möbius bond to the resulting boundary circle. Repeating this construction gives the other non-orientable surfaces. Similarly, a non-orientable surface is uniquely determined by the number of Möbius bonds.

Definition 1.1 (2-cell embedding)**.** *Let Γ be a graph embedded in a surface $\mathbb{S}$. The embedding is called a 2-cell embedding if every connected component of $\mathbb{S}\setminus\Gamma$ is homeomorphic to an open Euclidean disc. These components are called the faces.*

Definition 1.2 (Map)**.** *Let Γ be a graph with vertex set V and edge set E. Assume that Γ has a 2-cell embedding on a surface $\mathbb{S}$, which decomposes $\mathbb{S}$ into a face set F. Denote the map by $\mathcal{M} = (V, E, F)$, where $\mathbb{S}$ is called the supporting surface, and Γ is called the underlying graph.*

The following are examples of maps:

- The five regular polyhedrons are maps on the sphere.
- $K_1^{(2)}$ embedded in a torus decomposes the torus into one single face, as Figure 1, where $K_1^{(2)}$ is a single vertex graph with two loops.
- K_4 embedded in a projective plane decomposes the plane into 3 faces;

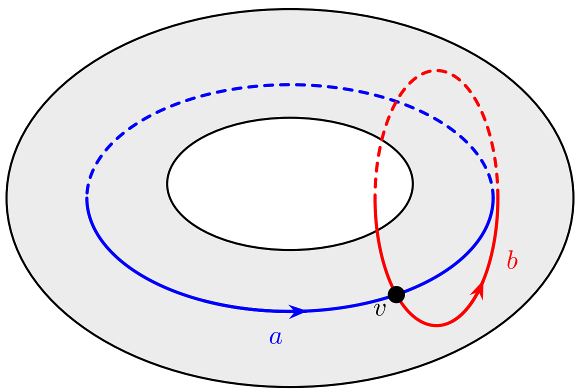

Figure 1: A 2-cell embedding of $K_1^{(2)}$ in the torus. The loop a goes around the large circle of the torus, while b goes around a cross-section.

The embedding of K_4 in the projective plane has the following three faces: $F_1 = 12341$, $F_2 = 14231$, $F_3 = 12431$. Let $F = F_1, F_2, F_3$ be the face set. We observe that each edge occurs exactly twice in the boundaries of these three faces, and the face corners around each vertex form a complete cyclic order. Hence, these three faces define a 2-cell embedding of K_4.

Definition 1.3 (Dual map)**.** *The dual map of a map $\mathcal{M} = (V, E, F)$ is the map $M^* = (F, E, V)$.*

Note that $K_3^{(2)}$ embedded in a projective plane decomposes the plane into 4 faces, as the dual map of $\mathcal{M} = (V(K_4), E(K_4), F)$, where $K_3^{(2)}$ is a complete graph on three vertices with 2 parallel edges joining each pair of distinct vertices.

Definition 1.4 (Automorphism)**.** *Let $\mathcal{M} = (V, E, F)$. A permutation of the set $V \cup E \cup F$ is called an automorphism of $\mathcal{M}$ if it preserves V, E, F and the incidence relation between vertices, edges and faces. All automorphisms of $\mathcal{M}$ form the automorphism group* $\operatorname{Aut} \mathcal{M}$.

Let $\mathcal{M} = (V, E, F)$ be a map, and let $G \leq \mathrm{Aut}\,\mathcal{M}$. Let e be an edge with endpoints α and β, and let f and f' be the faces incident with e.

In this course, we always assume that $|V|, |F| \geq 2$ and that the two sides of every edge are distinct faces. An incident triple (α, e, f) consisting of a vertex, an edge, and a face is called a *flag*. Note that there are four flags incident with e: (α, e, f), (α, e, f'), (β, e, f), (β, e, f'). Let $\mathcal{F}$ be the set of all flags in the map $\mathcal{M}$.

We set $G_\alpha = \{g \in G : \alpha^g = \alpha\}$, $G_e = \{g \in G : e^g = e\}$, $G_f = \{g \in G : f^g = f\}$ and $E(\alpha) = \{e \in E : e \text{ is incident with } \alpha\}$.

Claim 1.5. *If G_α is faithful on $E(\alpha)$, then G_α is cyclic or dihedral; If G_e is faithful on the four flags incident with e, then $G_e \leq D_4 \cong C_2 \times C_2$.*

Proof. Let α be a vertex of valency d. Consider a homomorphism $\rho\colon G_\alpha \to \mathrm{Sym}(E(\alpha))$. Since every automorphism fixing α must preserve or reverse the cyclic order of the edges incident with α, for any $g \in G_\alpha$, the permutation $\rho(g)$ is either a rotation or a reflection of $E(\alpha)$. Hence $\rho(G_\alpha) \leq D_{2d}$, where D_{2d} denotes the dihedral permutation group on $E(\alpha)$. Moreover, G_α is faithful on $E(\alpha)$. Thus, we have $\ker \rho = \mathbf{1}$, and then ρ is injective. Hence, $G_\alpha \cong \rho(G_\alpha) \leq D_{2d}$.

Since every subgroup of a dihedral group is either cyclic or dihedral, it follows that G_α is cyclic or dihedral. Similarly, since an incidence-preserving permutation of these four flags can only interchange the two endpoints of e, interchange the two faces incident with e, or do both, the induced permutation group is $D_4 \cong C_2 \times C_2$. And by faithfully on the four flags, We have $G_e \leq D_4 \cong C_2 \times C_2$. □

Definition 1.6 (Transitivity)**.** *If $G \leq \mathrm{Aut}\,\mathcal{M}$ is transitive on E, then M is called edge-transitive; similarly, we can define vertex-transitive, and flag-transitive.*

2 Regular map

Definition 2.1 (Regular map)**.** *Let $\mathcal{M} = (V, E, F)$ be a flag-transitive map. If an automorphism of $\mathcal{M}$ fixes a flag, then it fixes everything of the map, and hence is the identity(flag-stabilizer is trivial). Such a map $\mathcal{M}$ is called a regular map.*

Fact 2.2. *Let $\mathcal{M} = (V, E, F)$ be a regular map, and let $\mathcal{F}$ be the set of all flags in the map $\mathcal{M}$. Then for any two distinct $F_1, F_2 \in \mathcal{F}$, there exists unique automorphism $g \in \mathrm{Aut}\,\mathcal{M}$ such that $F_1^g = F_2$.*

Proof. Existence follows from flag-transitivity. It remains to prove the uniqueness. Suppose not. Assume that there are two distinct automorphisms $g_1, g_2 \in \mathrm{Aut}\,\mathcal{M}$ such that $F_1^{g_1} = F_2$ and $F_1^{g_2} = F_2$. Then we have $F_1 = F_1^{{g_1}^{-1} g_2}$. Since the flag-stabilizer is trivial, we know ${g_1}^{-1} g_2 = \mathbf{1}$, and hence $g_1 = g_2$, a contradiction. □

Fact 2.3. *Let $\mathcal{M} = (V, E, F)$ be a regular map, and let $\mathcal{F}$ be the set of all flags in the map $\mathcal{M}$. Then we have $|\operatorname{Aut}\mathcal{M}| = |\mathcal{F}| = 4|E|$.*

Proof. Indeed, we can construct a mapping between $\operatorname{Aut}\mathcal{M}$ and $\mathcal{F}$ by mapping $g \in \operatorname{Aut}\mathcal{M}$ to $\Phi^g \in \mathcal{F}$, where Φ is a fixed flag. By Fact 2.2, we know the mapping is bijective. Hence, we have $|\operatorname{Aut}\mathcal{M}| = |\mathcal{F}|$. Moreover, since an edge is incident four flags, the result holds. $\square$

Lemma 2.4. *Let $\mathcal{M} = (V, E, F)$ be a regular map, and $G = \operatorname{Aut}\mathcal{M}$. Fixing a flag (α, e, f), there exist involutions x, y, z with $xz = zx$ such that $G = \langle x, y, z\rangle$, $G_\alpha = \langle x, y\rangle$, $G_f = \langle y, z\rangle$ and $G_e = \langle x, z\rangle \cong D_4$.*

Proof. Let $\Phi = (\alpha, e, f)$ be a fixed flag. Let β be the other endpoint of e, let f' be the other face incident with e, and let e' be the other edge incident with the tube (α, f). Since $\mathcal{M}$ is regular, we obtain that $\mathcal{M}$ is flag-transitive and the flag-stabilizer is trivial. Hence, there exist unique elements $x, y, z \in G$ such that $\Phi^x = (\alpha, e, f')$, $\Phi^y = (\alpha, e', f)$, and $\Phi^z = (\beta, e, f)$, respectively. Clearly, x, y, z are involutions, that is, $x^2 = y^2 = z^2 = \mathbf{1}$. Moreover, it is easy to check that $\Phi^{xz} = \Phi^{zx} = (\beta, e, f')$. Then, it follows that $\Phi^{(xz)^{-1}zx} = \Phi$. Since the flag-stabilizer is trivial, we get $(xz)^{-1}zx = \mathbf{1}$. Therefore $xz = zx$.

Next, we prove that $G = \langle x, y, z\rangle$. Starting from Φ, any flag of $\mathcal{M}$ can be reached by a sequence of adjacent flags, where at each step one changes exactly one of the vertex, edge, or face. These three types of changes are induced by z, y, and x, respectively. Hence every flag is of the form Φ^w for some $w \in \langle x, y, z\rangle$. On the other hand, for every $g \in G$, Φ^g is a flag. Thus there exists $w \in \langle x, y, z\rangle$ such that $\Phi^g = \Phi^w$. Since the flag-stabilizer is trivial, we have $g = w$. Therefore $G = \langle x, y, z\rangle$. Since both x and y fix α, we have $\langle x, y\rangle \le G_\alpha$. Conversely, if $g \in G_\alpha$, then Φ^g is a flag incident with α. Any two flags incident with α can be connected by successively changing the incident face or the incident edge while keeping α fixed, and these two operations are induced by x and y. Hence there exists $w \in \langle x, y\rangle$ such that $\Phi^g = \Phi^w$. By the triviality of the flag stabilizer, $g = w$, and thus $G_\alpha = \langle x, y\rangle$. Similarly, we have $G_f = \langle y, z\rangle$ and $G_e = \langle x, z\rangle$. More precisely, by $xz = zx$, it is obviously that $G_e = \{1, x, z, xz\} = \langle 1, x\rangle \times \langle 1, z\rangle = C_2 \times C_2 \cong D_4$. $\square$

In the following, we conversely construct a map from a given group. In this course, for any group G and $H \le G$, we denote $[G : H] = \{Hg : g \in G\}$.

Definition 2.5. *Given a group $G = \langle x, y, z\rangle$, where x, y, z are involutions with $xz = zx$, define an incidence configuration (V, E, F), where*

$$V := [G : \langle x, y\rangle],\ E := [G : \langle x, z\rangle],\ F = [G : \langle y, z\rangle]$$

such that the incidence relation between $\alpha \in V$, $e \in E$ and $f \in F$ are defined by non-empty intersection of the corresponding cosets.

To show that the definition is well-defined, we first prove that the incidence relation is reasonable.

Claim 2.6. $\Gamma = (V, E)$ *is a graph, and every edge is incident with* $\langle x, z\rangle g$ *is incident with* $\langle y, z\rangle g$ *and* $\langle y, z\rangle xg$.

Proof. For any $g \in G$, we have an edge $\langle x, z\rangle g$. More precisely, the group $\langle x, z\rangle = \{1, x, z, xz\}$ because x, z are involutions with $xz = zx$, and then $\langle x, z\rangle g = \{g, xg, zg, xzg\}$. Since $[G : \langle x, y\rangle]$ partition the group $G = \langle x, y, z\rangle$, for any element $t \in \langle x, z\rangle g$, there exists an element $g' \in G$ such that $t \in \langle x, y\rangle g'$. Thus, by the definition, the vertex $\langle x, y\rangle g'$ is incident with the edge $\langle x, z\rangle g$. Moreover, we observe that

- if $t = g$ or $t = xg$, then $g' \in \langle x, y\rangle g$ and further we have $\langle x, y\rangle g' = \langle x, y\rangle g$;
- if if $t = zg$ or $t = xzg$, then $g' \in \langle x, y\rangle zg$ and further we have $\langle x, y\rangle g' = \langle x, y\rangle zg$.

Hence, every edge $\langle x, z\rangle g$ has two endpoints $\langle x, y\rangle g$ and $\langle x, y\rangle zg$. In the same way, we can show that every edge $\langle x, z\rangle g$ is incident with $\langle y, z\rangle g$ and $\langle y, z\rangle xg$. □

By the definitions of V, E and F as right coset spaces, G acts transitively on V, E and F by right multiplication. Moreover, this action preserves the incidence relation, since non-empty intersections of cosets are preserved under right multiplication.

Claim 2.7. *The edge* $\langle x, z\rangle$ *is contained in some cycle in* Γ*, and the cycle is a boundary of* $\langle y, z\rangle$.

Proof. Let $\rho = yz \in \langle y, z\rangle$. For integer $0 \leq i \leq |\rho| - 1$, we take edge $\langle x, z\rangle\rho^i$. Since $\rho^i \in \langle x, z\rangle\rho^i \cap \langle y, z\rangle$, then every edge $\langle x, z\rangle\rho^i$ is incident with the face $\langle y, z\rangle$. By Claim 2.6, we know $\langle x, z\rangle g$ has two endpoints $\langle x, y\rangle g$ and $\langle x, y\rangle zg$. Thus, we have $\langle x, z\rangle\rho^i$ has two endpoints $\langle x, y\rangle\rho^i$ and $\langle x, y\rangle\rho^{i+1}$. Hence,

$$\langle x, y\rangle, \langle x, z\rangle, \langle x, y\rangle\rho, \langle x, z\rangle\rho, \langle x, y\rangle\rho^2, \cdots, \langle x, y\rangle\rho^{|\rho|-1}, \langle x, z\rangle\rho^{|\rho|-1}, \langle x, y\rangle\rho^{|\rho|} = \langle x, y\rangle$$

forms a cycle. Furthermore, we claim that these edges are all the incidence with $\langle y, z\rangle$. Since $\rho = yz$ and y, z are involutions, we can identify $\langle y, z\rangle = \langle \rho, z : \rho^{|\rho|} = \mathbf{1}, z^2 = \mathbf{1}, z\rho z = \rho^{-1}\rangle$ as a dihedral group. Then every element of $\langle y, z\rangle$ has the form $\rho^i, z\rho^i$ for any integer $0 \leq i \leq |\rho| - 1$. If $\rho^i \in \langle x, z\rangle g \cap \langle y, z\rangle$ for some $g \in G$, then we have $\langle x, z\rangle g = \langle x, z\rangle\rho^i$; if $z\rho^i \in \langle x, z\rangle g \cap \langle y, z\rangle$ for some $g \in G$, then we also have $\langle x, z\rangle g = \langle x, z\rangle\rho^i$; Hence, the cycle is a boundary of $\langle y, z\rangle$. □

Given a graph $\Gamma = (V, E)$, a collection $\mathcal{C}$ of cycles of Γ is called a *cycle-double-cover*(a CDC) if each edge of Γ appears in exactly two cycles in $\mathcal{C}$, or equivalently, each edge of Γ is covered two times by the cycles in $\mathcal{C}$.

Since G acts transitively on F while preserving the incidence relation, it follows from Claim 2.7 that the edges incident with each element in F form a cycle. Moreover, by Claim 2.6, each edge is incident with exactly two elements in F since assume $|F| \geq 2$. Therefore, the collection $\mathcal{C} = \{\mathcal{C}^g : g \in G\}$ forms a cycle-double-cover of Γ. However, in general, a cycle-double-cover does not necessarily form the boundary cycles of a map.

In the following, we aim to identify each cycle in $\mathcal{C}$ with an open Euclidean disc. To this end, we first introduce the notions of a *combinatorial map* and a *Riemann surface.*

Definition 2.8 (Combinatorial map)**.** *Let $G = \langle x, y, z\rangle$, where x, y, z are involutions with $xz = zx$. We call the graph $\Gamma = (V, E)$ constructed above the* coset graph *of G with respect to $\langle x, y\rangle$ and $\langle x, z\rangle$, and denote it by $\mathrm{Cos}(G, \langle x, y\rangle, \langle x, z\rangle)$.*

Let $\mathcal{C} = \{C^g : g \in G\}$ be the collection of cycles constructed above. Then the incidence configuration $(V, E, \mathcal{C})$ is called a combinatorial map*, and is denoted by $\mathcal{M}(G, x, y, z) = (V, E, \mathcal{C})$.*

Definition 2.9 (Riemann surface)**.** *A Riemann surface is a topological space of which sufficient small neighborhoods of each point are Euclidean discs.*

Lemma 2.10. *Given $\mathcal{M}(G, x, y, z) = (V, E, \mathcal{C})$, identifying each cycle in $\mathcal{C}$ with an open Euclidean disc defines*

- *a closed Riemann surface $\mathbb{S}$, and*
- *a 2-cell embedding of $\Gamma = (V, E)$ in $\mathbb{S}$.*

Let F denote the set of discs corresponding to the cycles in $\mathcal{C}$. Then $\mathcal{M} = (V, E, F)$ is a regular map with $\mathrm{Aut}\,\mathcal{M} = G$.

Proof. First, since G acts transitively on V, E, F and preserves the incidence relation, we have $G \leq \mathrm{Aut}\,\mathcal{M}$.

Next, we show that $\mathrm{Aut}\,\mathcal{M}$ has trivial flag-stabilizer. Let $\Phi = (\alpha, e, f)$ be a flag and let $g \in \mathrm{Aut}\,\mathcal{M}_\Phi$. Then g fixes α, e and f. Since e has exactly two endpoints and is incident with exactly two faces, g also fixes the other endpoint of e and the other face incident with e. Moreover, since the boundary of f is a cycle, the other edge incident with α on the boundary of f is also fixed by g. Hence all flags adjacent to Φ are fixed by g. Repeating this argument, and using the connectedness of the map, we conclude that g fixes every flag, and hence every vertex, edge and face of $\mathcal{M}$. Thus $g = \mathbf{1}$, and therefore $\mathrm{Aut}\,\mathcal{M}_\Phi = \mathbf{1}$.

Now, we prove that G is transitive on the set of flags. Since $G \leq \mathrm{Aut}\,\mathcal{M}$, if G is transitive on the set of flags, then $\mathrm{Aut}\,\mathcal{M}$ is transitive on the set of flags. Together with the fact that the flag stabilizer in $\mathrm{Aut}\,\mathcal{M}$ is trivial, this implies that $\mathcal{M} = (V, E, F)$ is a regular map. Let $\mathcal{F}$ be the set of flags of $\mathcal{M}$, and fix the flag $\Phi = (\langle x, y\rangle, \langle x, z\rangle, \langle y, z\rangle)$. For any edge $\langle x, z\rangle g$ with $g \in G$, its two incident vertices are $\langle x, y\rangle g$ and $\langle x, y\rangle zg$, and its two incident faces are $\langle y, z\rangle g$ and $\langle y, z\rangle xg$. Hence, the four flags incident with $\langle x, z\rangle g$ are Φ^g, Φ^{xg}, Φ^{zg} and Φ^{xzg}. Therefore, every flag lies in the G-orbit of Φ, and hence G is transitive on $\mathcal{F}$.

Let $h \in \mathrm{Aut}\,\mathcal{M}$ and fix a flag $\Phi \in \mathcal{F}$. Suppose that $\Phi^h = \Phi'$. Since G acts transitively on $\mathcal{F}$, there exists $g \in G$ such that $\Phi^g = \Phi'$. Hence $\Phi^h = \Phi^g$. Since the flag stabilizer in $\mathrm{Aut}\,\mathcal{M}$ is trivial, we have $hg^{-1} = 1$, and hence $h = g \in G$. Therefore, $\mathrm{Aut}\,\mathcal{M} = G$. □

Conversely, we have the following theorem.

Theorem 2.11. *Each regular map is isomorphic to a regular coset map.*

2.1 A class of edge-regular maps

Let $G = \langle b, w\rangle$ be a 2-generated group. We define a graph $\Gamma = (V, E)$, where $V = [G : \langle b\rangle] \cup [G : \langle w\rangle]$, $E = G$. For any vertex in Γ, say $\langle b\rangle x$ or $\langle w\rangle y$, it is incident with the edge $g \in G$ if and only if $g \in \langle b\rangle x$ or $g \in \langle w\rangle y$. The graph is called a *coset graph*, denoted by $\mathrm{Cos}(G, \langle b\rangle, \langle w\rangle)$.

Note that this graph is bipartite with two vertex sets $[G : \langle b\rangle]$ and $[G : \langle w\rangle]$ by the partition property of right cosets. Now, we show that the graph Γ is has a cycle-double-cover. An *arc* is an incident pair (α, e) be a vertex α and an edge e. Given a group $G = \langle b, w\rangle$, define a cycle $C(b, w)$ of graph Γ by starting with the 2-arc $(\langle w\rangle, 1, \langle b\rangle, b^{-1})$, repeatedly right multiply by powers of $(bw)^{-1}$. This gives

$$\langle w\rangle, 1, \langle b\rangle, b^{-1}, \ldots, \langle w\rangle(bw)^{-i}, (bw)^{-i}, \langle b\rangle(bw)^{-i}, b^{-1}(bw)^{-i}, \ldots, \langle b\rangle w, w, \langle w\rangle,$$

We denote this cycle by $C(b, w)$.

Since the right multiplication of G induces a group of automorphisms of Γ, for any $g \in G$, the image $C(b, w)g$ of $C(b, w)$ under g is also a directed cycle of Γ, which starts with the 2-arc $(\langle w\rangle g, g, \langle b\rangle g, b^{-1}g)$, repeatedly right multiply by powers of $(bw)^{-1}g$.

Let $\mathcal{C} = \{C(b, w)g \mid g \in G\}$ be the set of cycles of $\Gamma = \mathrm{Cos}(G, \langle b\rangle, \langle w\rangle)$. Then, we have the following lemma.

Lemma 2.12. *The cycles in $\mathcal{C}$ form a cycle-double-cover of Γ.*

Proof. Let $g \in E(\Gamma) = G$ be an arbitrary edge. Since the base cycle $C(b, w)$ contains the edges 1 and b^{-1}, the edge g occurs in both $C(b, w)g$ and $C(b, w)bg$. More precisely, in $C(b, w)g$, the edge g is traversed from $\langle w\rangle g$ to $\langle b\rangle g$, while in $C(b, w)bg$, it is traversed from $\langle b\rangle g$ to $\langle w\rangle g$. Thus these two occurrences correspond to the two sides of the edge g.

It remains to show that these are the only two occurrences of g. Suppose that g lies in $C(b, w)a$ for some $a \in G$. Since the edges of $C(b, w)$ are of the form $(bw)^{-i}$ or $b^{-1}(bw)^{-i}$, either $g = (bw)^{-i}a$ or $g = b^{-1}(bw)^{-i}a$ for some i. In the first case, $a = (bw)^i g$, and hence $C(b, w)a = C(b, w)g$, since $C(b, w)$ is invariant under right multiplication by $\langle bw\rangle$. In the second case, $a = (bw)^i bg$, and similarly $C(b, w)a = C(b, w)bg$.

Therefore, every edge of Γ occurs exactly twice in the directed cycle collection $\mathcal{C}$. Hence, $\mathcal{C}$ forms a cycle-double-cover of Γ. □

Since right multiplication by G induces automorphisms of Γ, for any two distinct edges $a, c \in E(\Gamma)$, there exists $g = a^{-1}c \in G$ such that $a^g = a^{a^{-1}c} = c$. Thus G acts transitively on $E(\Gamma)$. Moreover, if $a^g = a$, then $ag = a$, and hence $g = 1$. Therefore, G acts regularly on $E(\Gamma)$. Arguing as for regular maps yields that the graph Γ associated with the cycle set C gives rise to an edge-regular map.

3 Three types of geometries

3.1 Spherical, Euclidean and hyperbolic

We first recall the classification theorem for closed surfaces. A fundamental theorem in topology states that every connected closed surface is either orientable or non-orientable, and is determined up to homeomorphism by its orientability and genus.

Moreover, the Euler characteristic of a connected closed surface $\mathbb{S}$ is determined by its genus. More precisely, $\chi(\mathbb{S}) = 2 - 2g(\mathbb{S})$ if $\mathbb{S}$ is orientable, while $\chi(\mathbb{S}) = 2 - g(\mathbb{S})$ if $\mathbb{S}$ is non-orientable.

Let $G = \langle b, w\rangle$ be a finite group, and let $\mathcal{M} = \mathcal{M}(G, b, w)$ be the corresponding regular dessin. Set $m = |b|$, $n = |w|$, and $\ell = |bw|$. By the coset construction, the two parts of the vertex set have sizes $|G|/m$ and $|G|/n$, respectively. Moreover, the edge set is identified with G, and the faces correspond to the right cosets of $\langle bw\rangle$. Hence $|V| = |G|/m + |G|/n$, $|E| = |G|$, and $|F| = |G|/\ell$. It follows that the Euler characteristic of the supporting surface $\mathbb{S}$ is $\chi(\mathbb{S}) = |V| - |E| + |F| = |G|(1/m + 1/n + 1/\ell - 1)$.

The parameters m, n, ℓ also have a geometric interpretation. The regular dessin can be subdivided into fundamental triangles whose angles are π/m, π/n, and π/ℓ. Thus the geometry of the corresponding triangle is determined by the value of $1/m+1/n+1/\ell$. Accordingly, we distinguish the following three cases. In these three cases, we assume $1 \le m \le n \le \ell$ and $\ell \ge 2$.

Spherical case. Suppose that $1/m + 1/n + 1/\ell > 1$. Then $\chi(\mathbb{S}) > 0$. Since the surface associated with a regular dessin is orientable, $\chi(\mathbb{S}) = 2-2g(\mathbb{S})$, and hence $g(\mathbb{S}) = 0$ and $\chi(\mathbb{S}) = 2$. Thus $\mathbb{S}$ is the sphere.

Since $1 \le m \le n \le \ell$ and $\ell \ge 2$, the possible triples are $(1, n, n)$, $(2, 2, \ell)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$. The corresponding spherical triangle groups and dessins include $T_{1,n,n} \cong C_n$ with dessin $K_{1,n}$, $T_{2,2,\ell} \cong D_{2\ell}$ with a cycle as the underlying graph, $T_{2,3,3} \cong A_4$ corresponding to the subdivision of the tetrahedron, $T_{2,3,4} \cong S_4$ corresponding to the subdivision of the cube (or dually the octahedron), and $T_{2,3,5} \cong A_5$ corresponding to the subdivision of the dodecahedron (or dually the icosahedron).

Euclidean case. Suppose that $1/m+1/n+1/\ell = 1$. Then $\chi(\mathbb{S}) = 0$, and hence $g(\mathbb{S}) = 1$. Therefore, the supporting surface is a torus. Since $1 \le m \le n \le \ell$ and $\ell \ge 2$, the possible triples are $(2, 3, 6)$, $(2, 4, 4)$, and $(3, 3, 3)$.

In this case the fundamental triangle has angle sum π, and hence the associated triangle group acts on the Euclidean plane. The Euclidean triangle groups have the form $T_{m,n,\ell} \cong \mathbb{Z}^2 : \mathbb{Z}_\ell$, where $\ell \in \{3, 4, 6\}$. In particular, these triangle groups are infinite and solvable. Finite regular dessins of Euclidean type can be obtained from periodic Euclidean tessellations by taking suitable quotients, giving finite maps on the torus.

Hyperbolic case. Suppose that $1/m + 1/n + 1/\ell < 1$. Then $\chi(\mathbb{S}) < 0$, and hence $g(\mathbb{S}) \geq 2$. Note that the surface associated with a regular dessin is orientable. In this case the fundamental triangle has angle sum less than π, and the corresponding tessellation lies in the hyperbolic plane.

The associated triangle group $T_{m,n,\ell}$ is infinite and non-solvable. Nevertheless, such triangle groups may have finite quotients, and these finite quotients give rise to regular dessins on closed orientable surfaces of genus at least two.

3.2 Regular complete maps

Let $\mathcal{M}$ be a regular map with underlying graph $\Gamma = K_n^{(\lambda)}$, where $K_n^{(\lambda)}$ denotes the complete graph on n vertices with λ parallel edges joining each pair of distinct vertices. Let $G = \mathrm{Aut}\,\mathcal{M}$.

We first introduce the notion of 2-transitivity on the vertex set $V(\Gamma)$. For any two ordered pairs of distinct vertices (α_1, β_1) and (α_2, β_2) in $V(\Gamma)$, if there always exists an automorphism $g \in G$ such that $\alpha_1^g = \alpha_2$ and $\beta_1^g = \beta_2$, then we say that G is 2-transitive on $V(\Gamma)$.

Now, we show that G is 2-transitive on $V(\Gamma)$. Since $\mathcal{M}$ is regular, G is flag-transitive. In particular, for any vertex α, the stabilizer G_α acts transitively on the neighbors of α. Since Γ is complete, every vertex different from α is adjacent to α, and hence G_α is transitive on $V(\Gamma) \setminus \{\alpha\}$. Together with the transitivity of G on $V(\Gamma)$, this implies that G is 2-transitive on $V(\Gamma)$.

Let K be the kernel of the action of G on $V(\Gamma)$, that is, $K = \{g \in G : v^g = v \text{ for every } v \in V(\Gamma)\}$. Then G/K acts faithfully and 2-transitively on $V(\Gamma)$. Moreover, since $K \leq G_\alpha$, the stabilizer of α in G/K is $(G/K)_\alpha \cong G_\alpha/K$. Since M is regular, by Lemma 2.4, $G_\alpha = \langle x, y\rangle$, where x and y are involutions. Hence G_α is a dihedral group.

By the classification of 2-transitive permutation groups, the possible quotients are $G/K \cong S_3$, S_4, or A_5, with point stabilizers isomorphic to D_2, D_6, or D_{10}, respectively. Hence, by the orbit–stabilizer theorem, the corresponding degrees are $|V| = |S_3|/|D_2| = 3$, $|V| = |S_4|/|D_6| = 4$, and $|V| = |A_5|/|D_{10}| = 6$, respectively. Thus the relevant complete graphs occur on 3, 4, or 6 vertices.

We now examine the kernel K for complete graphs. The elements of K fix every vertex and may only permute the λ parallel edges between pairs of vertices. Since the local action is cyclic or dihedral, K is of cyclic or dihedral subgroup.

Suppose first that $\Gamma = K_2^{(\lambda)}$. Since there are only two vertices, the induced permutation group on V is $G/K \cong S_2$. Moreover, $K \cong \mathbb{Z}_\lambda$ or D_λ. Hence G is an extension of $\mathbb{Z}_\lambda$ or D_λ by $\mathbb{Z}_2$, that is, $G \cong \mathbb{Z}_\lambda.\mathbb{Z}_2$ or $D_\lambda.\mathbb{Z}_2$. Since G is generated by three involutions, the possible groups reduce to $G \cong D_{2\lambda}$ or $S_2 \times D_\lambda$.

Next suppose that $\Gamma = K_3^{(\lambda)}$. Then the induced action on the three vertices gives $G/K \cong S_3$, while again $K \cong \mathbb{Z}_\lambda$ or D_λ. Hence G is an extension of K by S_3. In the cyclic case one may write $G \cong \mathbb{Z}_\lambda.S_3 = (\mathbb{Z}_\lambda \times \mathbb{Z}_3).\mathbb{Z}_2$ or $\mathbb{Z}_{3\lambda}.\mathbb{Z}_2$.

For $n \geq 4$, the classification above leaves only the cases $n = 4$ and $n = 6$. More precisely, we have the following theorem.

Theorem 3.1. *Let $\Gamma = (V, E)$ be a complete multi-graph $K_n^{(\lambda)}$ of order $n \geq 4$. Let $\mathcal{M}$ be a flag-regular embedding of Γ, and let $G = \mathrm{Aut}\,\mathcal{M}$. Then either*

- *$\Gamma = K_4^{(\lambda)}$, and $G = 2^2{:}D_{6\lambda}$, or $(\mathbb{Z}_{\lambda\circ 8}).D_6$ with λ divisible by 4, or*
- *$\Gamma = K_6$ or $K_6^{(2)}$, and $G = A_5$ or $2 \times A_5$, respectively.*

4 Covering of groups, maps and surfaces

We first recall the triangle group $T_{m,n,\ell} = \langle b, w \mid b^m = w^n = (bw)^\ell = 1\rangle$, where $1/m + 1/n + 1/\ell \leq 1$. A natural problem is to characterize the finite quotients of $T_{m,n,\ell}$. In particular, for a given triple (m, n, ℓ), one may ask which finite groups arise as quotients of $T_{m,n,\ell}$.

Let $G = \langle b, w\rangle$ be a finite (m, n, ℓ)-group, where $m = |b|$, $n = |w|$, and $\ell = |bw|$. Associated with (G, b, w) are the coset graph $\mathrm{Cos}(G, \langle b\rangle, \langle w\rangle)$, the regular dessin $\mathcal{M}(G, b, w)$, and its supporting surface $\mathbb{S}$. The Euler characteristic of $\mathbb{S}$ is $\chi(\mathbb{S}) = |V|-|E|+|F| = |G|(1/|b|+1/|w|-1+1/|bw|) = |G|(1/m+1/n+1/\ell-1)$.

4.1 Smooth quotients of groups

Let $G = \langle b, w\rangle$ and let $\overline{G} = \langle \bar{b}, \overline{w}\rangle$ be a quotient of G. Thus there exists a surjective homomorphism $\pi : G \to \overline{G}$ such that $\pi(b) = \bar{b}$ and $\pi(w) = \overline{w}$. If $|b| = |\bar{b}| = m$, $|w| = |\overline{w}| = n$, and $|bw| = |\bar{b}\,\overline{w}| = \ell$, then $\overline{G}$ is called a *smooth quotient* of G, and G is called a *smooth cover* of $\overline{G}$.

The smoothness condition means that the parameters (m, n, ℓ) are preserved under the quotient.

4.2 Smooth coverings of maps

Let $\mathcal{M} = \mathcal{M}(G, b, w)$ and $\overline{\mathcal{M}} = \mathcal{M}(\overline{G}, \bar{b}, \overline{w})$. If $\overline{G}$ is a smooth quotient of G, then the two maps have the same bi-valency $(|b|, |w|)$ and the same face length $|bw|$. In this case, $\mathcal{M}$ is called a *smooth cover* of $\overline{\mathcal{M}}$.

Thus a smooth quotient at the group level induces a covering relation between the corresponding dessins, while preserving the local vertex degrees and face lengths.

4.3 Smooth coverings of surfaces

Let $\mathbb{S}$ and $\mathbb{S}'$ be closed orientable surfaces. If there exists a continuous map $\varphi : \mathbb{S} \to \mathbb{S}'$ satisfying every point of $\mathbb{S}'$ has exactly r preimages in $\mathbb{S}$, then $\mathbb{S}$ is said to be a *r-sheeted smooth covering* of $\mathbb{S}'$. Equivalently, locally around every point of $\mathbb{S}'$, the inverse image consists of r disjoint copies.

4.4 Coverings of groups, maps, and surfaces

The covering relations at the three levels are compatible. More precisely, the following conditions are equivalent:

1. G is a smooth cover of the quotient group $\overline{G}$;
2. $\mathcal{M}(G,b,w)$ is a smooth cover of $\mathcal{M}(\overline{G},\bar{b},\overline{w})$;
3. the supporting surface $\mathbb{S}$ is a smooth cover of the quotient surface $\mathbb{S}'$.

4.5 Euler characteristic under smooth coverings

Let N be the kernel of the quotient map $G \to \overline{G}$, so that $\overline{G} \cong G/N$ and $|G| = |N||\overline{G}|$. If $\mathcal{M}(G,b,w)$ is a smooth cover of $\mathcal{M}(\overline{G},\bar{b},\overline{w})$, then $|b| = |\bar{b}|$, $|w| = |\overline{w}|$, and $|bw| = |\bar{b}\,\overline{w}|$. Therefore, $\chi(\mathcal{M}(G,b,w)) = |G|(1/|b| + 1/|w| - 1 + 1/|bw|) = |N||\overline{G}|(1/|\bar{b}| + 1/|\overline{w}| - 1 + 1/|\bar{b}\,\overline{w}|) = |N|\chi(\mathcal{M}(\overline{G},\bar{b},\overline{w}))$. Thus, for the corresponding supporting surfaces, $\chi(\mathbb{S}) = |N|\chi(\mathbb{S}')$. In particular, if $\mathbb{S} \to \mathbb{S}'$ is an r-sheeted smooth covering, then $\chi(\mathbb{S}) = r\chi(\mathbb{S}')$.

Now, we give two examples to illustrate the theory of smooth coverings.

Example 1. Let $G = \langle b, w\rangle$ be a nonabelian p-group of order p^3 and exponent p. Then every nontrivial element of G has order p, and in particular $|b| = |w| = |bw| = p$. Let $N := Z(G) = G' \cong \mathbb{Z}_p$, and let $\mathcal{D} = \mathcal{D}(G,b,w)$.

Since $|G| = p^3$ and $|\langle b\rangle| = |\langle w\rangle| = p$, the two parts of the vertex set of $\mathcal{D}$ each contain p^2 vertices. Hence $\mathcal{D}$ has $2p^2$ vertices. Moreover, since every vertex is p-degree, $\mathcal{D}$ has p^3 edges. And since $|bw| = p$, the number of faces is $|G|/|\langle bw\rangle| = p^2$. Therefore, $\chi(\mathcal{D}) = 2p^2 - p^3 + p^2 = p^2(3-p)$.

Now consider the quotient group $\overline{G} = G/N$. Since $N = G'$, the quotient $\overline{G}$ is abelian. Moreover, $|\overline{G}| = |G|/|N| = p^2$. Since G has exponent p, so does $\overline{G}$. Hence $\overline{G} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let $\bar{b} = bN$ and $\overline{w} = wN$. Then $\overline{G} = \langle \bar{b}, \overline{w}\rangle$, and $|\bar{b}| = |\overline{w}| = |\bar{b}\,\overline{w}| = p$. Thus $\overline{G}$ is a smooth quotient of G. Let $\mathcal{D}_N = \mathcal{D}(\overline{G},\bar{b},\overline{w})$. Since $|\overline{G}| = p^2$ and $|\bar{b}| = |\overline{w}| = |\bar{b}\,\overline{w}| = p$, the dessin $\mathcal{D}_N$ has $2p$ vertices, p^2 edges, and p faces. Hence $\chi(\mathcal{D}_N) = 2p - p^2 + p = p(3-p)$.

Since $|N| = p$, the corresponding supporting surface $\mathbb{S}$ is a p-sheeted smooth cover of the quotient surface $\mathbb{S}'$, and their Euler characteristics satisfy $\chi(\mathbb{S}) = p\,\chi(\mathbb{S}')$. In this example, $\chi(\mathbb{S}) = p^2(3-p)$ and $\chi(\mathbb{S}') = p(3-p)$.

Example 2. In this example, the group and its generators are given explicitly, and the smoothness condition is reduced to determining the order of the product of the two generators.

Let $G = SL(2,p)$, where p is an odd prime, and let $N = Z(G) \cong \mathbb{Z}_2$. Then $\overline{G} = G/N = PSL(2,p)$. Let $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $G = \langle b, w\rangle$ and $|b| = |w| = p$. Hence the regular dessin $\mathcal{D} = \mathcal{D}(G,b,w)$ is of valency p.

Let $\bar{b} = bN$ and $\overline{w} = wN$ be the images of b and w in $\overline{G}$. Since $|\bar{b}| = |b| = p$ and $|\overline{w}| = |w| = p$, the quotient $G \to \overline{G}$ is smooth if and only if $|bw| = |\bar{b}\,\overline{w}|$.

Claim 4.1. *$|bw| = |\bar{b}\,\overline{w}|$ if and only if $|bw|$ is odd.*

Proof. Since $N = \{I, -I\}$, the order of bw decreases in the quotient only if $(bw)^k = -I$ for some positive integer $k < |bw|$.

Let $r = |bw|$. If r is odd, then $|\bar{b}\overline{w}| = r$. Suppose not. Let $k < r$ be a positive integer such that $(\bar{b}\overline{w})^k = N$. Since $N = \{I, -I\}$, $(bw)^k = I$ or $-I$. Then it follows that $(bw)^{2k} = I$. Since $r = |bw|$ and r is odd, we have $r|k$, a contradiction. Conversely, if $r = |bw|$ is even, then $(bw)^{r/2}$ has order 2. Since the unique element of order 2 in $SL(2,p)$ is $-I$, we have $(bw)^{r/2} = -I \in N$. Hence $(\bar{b}\,\overline{w})^{r/2} = N$, and so $|\bar{b}\,\overline{w}| = r/2 < r$.

Therefore, $G = \langle b, w\rangle$ is a smooth cover of $\overline{G} = \langle \bar{b}, \overline{w}\rangle$ if and only if $|bw|$ is odd. ☐

A direct calculation gives $bw = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Moreover, for every positive integer k, $(bw)^k = \begin{pmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1} \end{pmatrix}$, where $F_1 = F_2 = 1$ and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 3$. Thus the order of bw in $SL(2,p)$ can be studied through the divisibility properties of Fibonacci numbers modulo p. The following result given in D.Wall(1960), J,Vinson(1963) and A.Dishong and M.Renault(2018) determines the relevant parity of $|bw|$.

Lemma 4.2. *Let $p \geq 5$ be a prime. Then the following hold:*

1. *If $p = 5$, then $|bw|$ is 10.*
2. *If $p \equiv \pm 1 \pmod 5$, then $|bw|$ divides $(p-1)/2$. In particular, $|bw|$ is odd if $p \equiv 3 \pmod 4$.*
3. *If $p \equiv \pm 2 \pmod 5$, then $|bw|$ divides $p+1$, but $|bw|$ does not divide $(p+1)/2$. Hence $|bw|$ is even.*

Combining $p \equiv \pm 1 \pmod 5$ with $p \equiv 3 \pmod 4$, we obtain $|bw|$ is odd when $p \equiv 11$ or $19 \pmod{20}$. Hence the smoothness of the quotient can be described as follows.

Proposition 4.3. *Let $G = SL(2,p)$ and $\overline{G} = PSL(2,p)$. Let $\mathcal{D} = \mathcal{D}(G, b, w)$ and $\overline{\mathcal{D}} = \mathcal{D}(\overline{G}, \bar{b}, \overline{w})$. Then,*

1. *$\mathcal{D}$ is a smooth cover of $\overline{\mathcal{D}}$ when $p \equiv 11$ or $19 \pmod{20}$;*
2. *$\mathcal{D}$ is not a smooth cover of $\overline{\mathcal{D}}$ when $p \equiv \pm 2 \pmod 5$.*

Note that $(bw)^5 = \begin{pmatrix} 34 & 55 \\ 55 & 89 \end{pmatrix}$. If $p = 11$, then $(bw)^5 \equiv I \pmod{11}$, and hence $|bw| = 5$. Thus $\mathcal{D}(SL(2,11), b, w) \to \mathcal{D}(PSL(2,11), \bar{b}, \overline{w})$ is smooth. Since $|Z(SL(2,11))| = 2$, the corresponding covering is a smooth double covering. In particular, the supporting surface $\mathbb{S}_{409}$ is a smooth double cover of $\mathbb{S}_{205}$. On the other hand, if $p = 5$, then $|bw| = 10$, which is even. Hence $\mathcal{D}(SL(2,5), b, w) \to \mathcal{D}(PSL(2,5), \bar{b}, \overline{w})$ is not smooth. In this case, the corresponding surface map $\mathbb{S}_{31} \to \mathbb{S}_{13}$ has 12 branched points.

4.6 Face-quasiprimitive regular dessins

The preceding examples illustrate how smooth coverings can be studied for specific groups and generating pairs. We now turn to a more general structural question: if a regular dessin admits a smooth quotient, what can be said about the structure of its automorphism group?

To answer this question, we first introduce the notion of quasiprimitivity. A permutation group $G \leq \mathrm{Sym}(\Omega)$ is said to be *quasiprimitive* if every non-trivial normal subgroup of G is transitive. A regular dessin $\mathcal{D}$ is called *face-quasiprimitive* if $\mathrm{Aut}\,\mathcal{D}$ acts quasiprimitively on the set of faces.

Theorem 4.4 (Chen, Fan, Li, Zhu, 2026). *Let $\mathcal{D}$ be a regular dessin, let $G = \mathrm{Aut}\,\mathcal{D}$ be face-quasiprimitive, and let $N = \mathrm{soc}(G)$. If $\mathcal{D}$ is a smooth cover of $\mathcal{D}_N$, then G/N is cyclic and $G = N : \mathbb{Z}_\ell$ is of type HA, TW, or AS.*

Conversely, in each of the following cases, G is a smooth cover of $\mathbb{Z}_\ell$:

1. *$G = N : \mathbb{Z}_\ell$ is of type HA, where $\ell \geq 3$;*
2. *$G = T \wr \mathbb{Z}_\ell = T^\ell : \mathbb{Z}_\ell$ is of type TW, where $\ell \geq 5$;*
3. *$G = \Sigma L(2, 2^\ell) = SL(2, 2^\ell) : \mathbb{Z}_\ell$, where $\ell \geq 5$ is a prime.*