Chromatic Thresholds and Related Problems

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Introduction

In this series of lectures, we will go over the classical chromatic threshold problems in extremal graph theory. At its core, this type of problems seeks for sufficient density conditions for graphs with forbidden substructure to have bounded complexity (in this particular case, bounded chromatic number). After covering the basics and classical results, we will then introduce some recent developments which connects this area to other areas such as combinatorial convexity, VC dimension theory, additive combinatorics etc. Prerequisite: basic knowledge of graph theory.

1. Turán problem and stability.
2. When does dense triangle-free graph have bounded chromatic number?
3. Connections to VC dimension theory.
4. Beyond chromatic threshold.
5. Chromatic thresholds for linear equations.

Contents

1 From Extremal Density to Chromatic Thresholds

1.1 Mantel's theorem

We start with the following problem.

Problem. Choose n distinct irrational numbers, where n is even. What is the maximum number of pairs whose sum is rational?

Let the chosen irrational numbers be the vertices of a graph G, where two vertices x and y are adjacent if $x + y \in \mathbb{Q}$. Then G is triangle-free. Indeed, if x, y, z form a triangle, then

$$2x = (x + y) + (x + z) - (y + z) \in \mathbb{Q},$$

which implies $x \in \mathbb{Q}$, a contradiction. Thus the pairs with rational sum correspond to the edges of a triangle-free graph. This leads naturally to Mantel's theorem.

Theorem 1.1 (Mantel, 1907 [11])**.** *Every n-vertex triangle-free graph G satisfies $e(G) \le \left\lfloor \frac{n^2}{4} \right\rfloor$. Moreover, equality is attained by the balanced complete bipartite graph.*

We shall give two proofs.

First proof. Choose a vertex v of maximum degree and let $\Delta = d(v)$. Since G is triangle-free, $N(v)$ is independent. Hence every edge of G has at least one endpoint in $V(G) \setminus N(v)$, and therefore

$$e(G) \le \sum_{u \in V(G)\setminus N(v)} d(u) \le (n - \Delta)\Delta \le \frac{n^2}{4}.$$ □

Second proof. We use Zykov symmetrization. Starting from $G_0 = G$, we modify the graph step by step to obtain a sequence $G_0, G_1, \ldots, G^*$, such that each G_i is triangle-free and $e(G_i) \ge e(G_{i-1})$.

Choose a vertex v of maximum degree, and let $\Delta := d(v)$, $B = N(v)$, and $A = V(G) \setminus B$. Since G is triangle-free, B is an independent set. For each $u \in A \setminus \{v\}$, replace u by a clone of v: delete all edges incident with u and join u to every vertex in B.

Obs 1: At every step the graph remains triangle-free, since B is independent.

Obs 2: At every step the number of edges does not decrease, since the current degree of u is at most Δ.

After all vertices in $A \setminus \{v\}$ have been symmetrized, every vertex of A has neighborhood B. Thus $G^* = K_{|A|,|B|} = K_{n-\Delta,\Delta}$. Consequently, $e(G) \le e(G^*) = \Delta(n - \Delta) \le \frac{n^2}{4}$. □

Returning to the initial question, the upper bound is therefore $n^2/4$. It is attained, for example, by choosing an irrational number x and taking $x + 1, \ldots, x + \frac{n}{2}$ and $-x + 1, \ldots, -x + \frac{n}{2}$. Every pair consisting of one number from each set has rational sum, giving exactly $\frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$ such pairs.

Remark 1.2 (Stability)**.** If G is an n-vertex triangle-free graph with $e(G) = \frac{n^2}{4} - o(n^2)$, then there is a partition $V(G) = A \cup B$ with $|A|, |B| = (\frac{1}{2} + o(1))n$ such that $|E(G) \triangle E(K_{A,B})| = o(n^2)$. Thus every nearly extremal triangle-free graph is close to the extremal construction.

Remark 1.3. Chromatic number is much more sensitive to local changes. In fact, triangle-free graphs can have arbitrarily large chromatic number; Mycielski's construction [12] gives a classical example.

Let H be such a graph on $m = o(n)$ vertices, and let G be the disjoint union of H and a balanced complete bipartite graph on the remaining $n - m$ vertices. Then G is triangle-free and $e(G) = n^2/4 - o(n^2)$, while $\chi(G) \ge \chi(H)$. Thus stability alone does not control the chromatic number.

1.2 The Andrásfai–Erdős–Sós threshold for bipartiteness

The previous example shows that a small set of low-degree vertices can still force the chromatic number to be large. This motivates imposing a minimum-degree condition. For triangle-free graphs, sufficiently large minimum degree in fact forces bipartiteness.

Theorem 1.4 (Andrásfai–Erdős–Sós, triangle-free case [2]). *If G is an n-vertex triangle-free graph with $\delta(G) > \frac{2n}{5}$, then G is bipartite.*

Remark 1.5. The constant $\frac{2}{5}$ is sharp. Indeed, a balanced blow-up of C_5 is triangle-free and non-bipartite, with minimum degree approximately $\frac{2n}{5}$.

Proof. Assume G is non-bipartite and let C be a shortest odd cycle. Since G is triangle-free, $|C| \geq 5$.

Obs 1: C has no chord, by the choice of C as a shortest odd cycle.

Obs 2: Every vertex $v \notin C$ has at most two neighbors on C. Otherwise, three neighbors of v on C would give either a triangle or a shorter odd cycle.

By double counting the edges incident with C,

$$|C|\delta(G) \leq \sum_{u \in C} d(u) \leq 2|C| + 2(n - |C|) = 2n.$$

Hence $\delta(G) \leq 2n/|C| \leq 2n/5$, a contradiction. □

1.3 Chromatic threshold

The AES theorem shows that a triangle-free graph with minimum degree greater than $2n/5$ must be bipartite. If we only ask for bounded chromatic number rather than bipartiteness, one may expect a smaller minimum-degree threshold. This motivates the following definition.

Definition 1.6. For a fixed graph H, its *chromatic threshold* is

$$\delta_\chi(H) := \inf \{\alpha : \ \exists C = C(H, \alpha) \text{ such that every } H\text{-free } G \text{ with } \delta(G) \geq \alpha|G| \text{ satisfies } \chi(G) \leq C\}.$$

For triangle-free graphs, the minimum-degree condition can indeed be weakened substantially when only bounded chromatic number is required.

Theorem 1.7 (Thomassen, 2002 [13]). *For every $\varepsilon > 0$, there exists $C = C(\varepsilon)$ such that every sufficiently large triangle-free graph G with $\delta(G) \geq (\frac{1}{3} + \varepsilon)|G|$ satisfies $\chi(G) \leq C$.*

Thus $\delta_\chi(K_3) \leq 1/3$. We first prove the matching lower bound, due to Hajnal, and then return to Thomassen's upper bound.

1.3.1 Hajnal's Kneser-graph construction: the lower bound $\delta_\chi(K_3) \geq 1/3$

To prove the lower bound, it suffices to construct triangle-free graphs with $\delta(G) \geq (\frac{1}{3} - o(1))n$ and arbitrarily large chromatic number.

Definition 1.8. The Kneser graph $\mathrm{KG}(m, k)$ has vertex set $\binom{[m]}{k}$, where two vertices are adjacent if and only if the corresponding k-sets are disjoint.

Theorem 1.9 (Lovász, 1978 [8]). $\chi(\mathrm{KG}(m, k)) = m - 2k + 2$.

Suppose that $1/n \ll 1/k \ll 1/t$ and set $m = 2k + t$. Let $K := \mathrm{KG}(m, k)$ and $M := |V(K)| = \binom{m}{k}$. Then $\chi(K) = t + 2$, and K is triangle-free since $m < 3k$. Assume that $3m \mid (n - M)$. Take a complete bipartite graph with parts A and B, where $|A| = \frac{2(n-M)}{3}$ and $|B| = \frac{n-M}{3}$. Partition $A = A_1 \cup \cdots \cup A_m$ into equal parts. For each $S \in V(K)$, join S to every vertex in A_i whenever $i \in S$. There are no edges between K and B.

The resulting graph is triangle-free. Indeed, K is triangle-free, and no triangle can contain a vertex of B. If $S, T \in K$ and $a \in A_i$ formed a triangle, then $S \cap T = \emptyset$, while $i \in S \cap T$, a

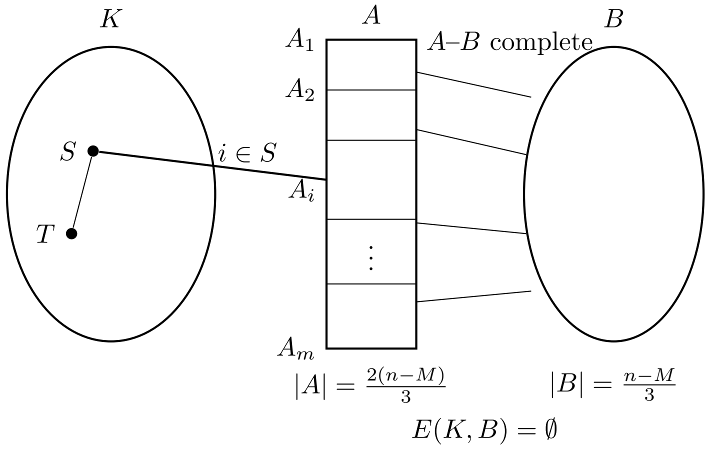

Figure 1: The Hajnal construction.

contradiction. Moreover, since K is an induced subgraph, the Hajnal graph has chromatic number at least $\chi(K) = t+2$.

It remains to check the minimum degree. Clearly, every vertex in B has degree $2(n-M)/3$, while every vertex in A has degree at least $(n-M)/3$. For each $S \in K$, $d(S) \geq k \cdot \frac{|A|}{m} = \frac{2k}{3m}(n-M)$. Since $M = o(n)$ and $m = 2k+t$ with $t \ll k$, we obtain $\delta(H) \geq \left(\frac{1}{3} - o(1)\right) n$.

1.3.2 Proof of Theorem 1.7

We first recall the regularity lemma. For two disjoint sets $A, B \subseteq V(G)$, let

$$d(A,B) := \frac{e(A,B)}{|A||B|}.$$

The pair (A,B) is *η-regular* if for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \eta|A|$ and $|B'| \geq \eta|B|$, we have $|d(A',B') - d(A,B)| \leq \eta$.

We will use the following degree form of the regularity lemma.

Lemma 1.10 (Regularity lemma, degree form)**.** *Let $\alpha, \varepsilon > 0$ with $\alpha + \varepsilon < 1$, and let $0 < \eta \ll d \ll \varepsilon$ and $k_0 \in \mathbb{N}$. Then there exists $K = K(\eta, k_0)$ such that every sufficiently large n-vertex graph G with $\delta(G) \geq (\alpha+\varepsilon)n$ admits a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$, where $k_0 \leq k \leq K$, $|V_0| \leq \eta n$, and $|V_1| = \cdots = |V_k|$. Moreover, the corresponding reduced graph R, where $ij \in E(R)$ if (V_i, V_j) is η-regular with density at least d, satisfies $\delta(R) \geq \left(\alpha + \frac{\varepsilon}{2}\right) k$.*

Proof of Theorem 1.7. Choose $0 < \eta \ll d \ll \varepsilon$ and apply the regularity lemma to G. Let $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ be the resulting partition, and let R be the reduced graph. Since $\delta(G) \geq (1/3+\varepsilon)n$, we have $\delta(R) \geq (1/3 + \varepsilon/2)k$.

Let $s := |V_1| = \cdots = |V_k|$. For each $I \subseteq [k]$, define

$$X_I := \{v \in V(G) : d(v, V_i) \geq d|V_i| \text{ if and only if } i \in I\}.$$

Then $\{X_I : I \subseteq [k]\}$ forms a partition of $V(G)$.

We first consider $|I| \geq 2k/3$. We claim that $X_I = \emptyset$. Indeed, suppose $v \in X_I$. Then I is an independent set in R. Otherwise, if $ij \in E(R[I])$, then $|N(v) \cap V_i|, |N(v) \cap V_j| \geq ds$. Since (V_i, V_j) is η-regular with density at least d, there is an edge between $N(v) \cap V_i$ and $N(v) \cap V_j$, which forms a triangle with v. Thus I is independent in R, and hence $|I| \leq k - \delta(R) < 2k/3$, a contradiction.

Now suppose $|I| < 2k/3$. We show that X_I is independent. Suppose that $x, y \in X_I$ and $xy \in E(G)$, and let $U := \bigcup_{i \in I} V_i$. Since $x, y \in X_I$, each has fewer than ds neighbors in every V_j with $j \notin I$. Thus each has at most $(k - |I|)ds + |V_0| \leq (d+\eta)n$ neighbors outside U. Hence

$$d(x,U), d(y,U) \geq \left(\frac{1}{3} + \varepsilon - d - \eta\right) n > \left(\frac{1}{3} + \frac{\varepsilon}{2}\right) n.$$

On the other hand, $|I| < 2k/3$ implies $|U| < 2n/3$. Therefore $d(x,U) + d(y,U) > |U|$, so x and y have a common neighbor in U. Together with $xy \in E(G)$, this gives a triangle, a contradiction. Thus X_I is independent.

Hence every nonempty X_I is independent. Since there are at most 2^k sets X_I and $k = O_\varepsilon(1)$, we obtain $\chi(G) \le 2^k = O_\varepsilon(1)$. □

Hajnal's construction and Theorem 1.7 together show that $\delta_\chi(K_3) = \frac{1}{3}$.

2 VC-dimension and Further Chromatic Thresholds

The regularity proof above is one way to obtain bounded coloring. We now present another approach based on transversals and VC-dimension.

2.1 Transversals and VC-dimension

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A *transversal* of $\mathcal{H}$ is a set $T \subseteq V$ such that $T \cap E \neq \emptyset$ for every $E \in \mathcal{E}$. The minimum size of a transversal is denoted by $\tau(\mathcal{H})$.

A *fractional transversal* is a function $f : V \to [0,1]$ such that $\sum_{v \in E} f(v) \ge 1$ for every $E \in \mathcal{E}$. Its fractional transversal number is

$$\tau^*(\mathcal{H}) := \min \left\{ \sum_{v \in V} f(v) : f : V \to [0,1],\ \sum_{v \in E} f(v) \ge 1 \text{ for every } E \in \mathcal{E} \right\}.$$

Clearly, $\tau^*(\mathcal{H}) \le \tau(\mathcal{H})$. In general, however, the gap between $\tau(\mathcal{H})$ and $\tau^*(\mathcal{H})$ can be arbitrarily large.

We now introduce VC-dimension, under which $\tau(\mathcal{H})$ can be bounded in terms of $\tau^*(\mathcal{H})$.

Definition 2.1. A set $S \subseteq V$ is *shattered* by $\mathcal{H}$ if for every $T \subseteq S$, there exists $E \in \mathcal{E}$ such that $E \cap S = T$. The *VC-dimension* of $\mathcal{H}$, denoted by $\mathrm{VC}(\mathcal{H})$, is the maximum size of a shattered set.

Theorem 2.2 (ε-net theorem [4])**.** *If* $\mathrm{VC}(\mathcal{H}) = d$, *then*

$$\tau(\mathcal{H}) \le 16d\,\tau^*(\mathcal{H}) \log(d\tau^*(\mathcal{H})).$$

Thus, for hypergraphs of bounded VC-dimension, a bounded fractional transversal number implies a bounded transversal number.

2.2 Bounded VC-dimension and coloring

We now apply the ε-net theorem to graph neighborhoods. Without any VC-dimension assumption, the triangle-free chromatic threshold is $1/3$. Under bounded VC-dimension, however, every positive linear minimum-degree condition already gives bounded chromatic number. For a graph G, let $\mathcal{N}_G$ denote the *neighborhood hypergraph* with vertex set $V(G)$ and edge set $\{N(v) : v \in V(G)\}$. We write $\mathrm{VC}(G) := \mathrm{VC}(\mathcal{N}_G)$.

Theorem 2.3. *Let* $\alpha > 0$ *and* $d \in \mathbb{N}$. *If* G *is an* n*-vertex triangle-free graph with* $\delta(G) \ge \alpha n$ *and* $\mathrm{VC}(G) \le d$, *then* $\chi(G) = O_{\alpha,d}(1)$.

Proof. Consider the neighborhood hypergraph $\mathcal{N}_G$. Define $f(v) = 1/(\alpha n)$ for every $v \in V(G)$. Since every neighborhood has size at least αn, f is a fractional transversal of $\mathcal{N}_G$, and hence $\tau^*(\mathcal{N}_G) \le 1/\alpha$.

By the ε-net theorem, $\mathcal{N}_G$ has a transversal $T = \{v_1, \dots, v_r\}$ with $r = O_{\alpha,d}(1)$. Since T meets every neighborhood, $V(G) = N(v_1) \cup \dots \cup N(v_r)$. Moreover, each $N(v_i)$ is independent since G is triangle-free. Therefore $\chi(G) \le r = O_{\alpha,d}(1)$. □

We next use this result to give an alternative proof of the upper bound $\delta_\chi(K_3) \le \frac{1}{3}$. We need two simple structural observations.

Lemma 2.4 (Brandt [15])**.** *Let* G *be a maximal triangle-free graph on* n *vertices. If* $\delta(G) > \frac{n}{3}$, *then* G *contains no induced copy of the cube* Q_3.

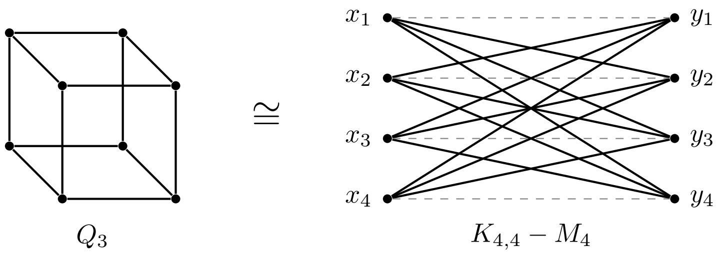

Figure 2: The cube Q_3.

Lemma 2.5. *If G is triangle-free and contains no induced copy of Q_3, then* $\mathrm{VC}(G) \leq 3$.

Proof. Recall that $Q_3 \cong K_{4,4} - M_4$; see Figure 2. Suppose instead that $\mathrm{VC}(G) \geq 4$, and let$X = \{x_1, x_2, x_3, x_4\}$ be shattered by the neighborhood hypergraph. Since X is shattered, there exists a vertex adjacent to all four vertices of X. As G is triangle-free, X must therefore be independent. For each $i \in [4]$, choose a vertex y_i such that $N(y_i) \cap X = X \setminus \{x_i\}$. The vertices $y_1, \dots, y_4$ are also independent.

Hence the graph induced by $\{x_1, \dots, x_4, y_1, \dots, y_4\}$ is $K_{4,4}$ with the perfect matching $x_i y_i$, $i \in [4]$, removed. This is precisely an induced copy of Q_3, a contradiction. □

We can now recover Thomassen's upper bound by the VC-dimension method. Let G be a triangle-free graph with $\delta(G) \geq \left(\frac{1}{3} + \varepsilon\right) n$. Add edges while preserving triangle-freeness until obtaining a maximal triangle-free supergraph G'. Then $\delta(G') \geq \delta(G) > \frac{n}{3}$. By Brandt's lemma, G' contains no induced copy of Q_3, and hence $\mathrm{VC}(G') \leq 3$ by the previous lemma. Applying the bounded VC-dimension coloring theorem to G' gives $\chi(G') = O_\varepsilon(1)$. Since $G \subseteq G'$, we also have $\chi(G) = O_\varepsilon(1)$. Therefore $\delta_\chi(K_3) \leq \frac{1}{3}$. Hajnal's construction and the argument above together show that $\delta_\chi(K_3) = \frac{1}{3}$.

2.3 C_5-free graphs

The bounded VC-dimension theorem shows that any positive linear minimum-degree condition suffices once the neighborhood hypergraph has bounded VC-dimension. For C_5-free graphs, no additional VC-dimension assumption is needed: the chromatic threshold itself is zero.

Theorem 2.6 (Thomassen [14]). *For every $\alpha > 0$, every sufficiently large n-vertex C_5-free graph G with $\delta(G) \geq \alpha n$ satisfies $\chi(G) \leq \frac{6}{\alpha}$. In particular, $\delta_\chi(C_5) = 0$.*

Proof. We first cover $V(G)$ by a bounded number of radius-two neighborhoods. Take a maximal collection $v_1, \dots, v_r$ of vertices with pairwise distance at least 3. Then the neighborhoods $N(v_i)$ are pairwise disjoint, and hence $r\alpha n \leq \sum_{i=1}^r |N(v_i)| \leq n$. Thus $r \leq 1/\alpha$. On the other hand, by maximality, every vertex of G has distance at most 2 from some v_i. Hence $V(G) = \bigcup_{i=1}^r N_{\leq 2}(v_i)$. We may therefore partition $V(G) = U_1 \cup \cdots \cup U_r$ such that $U_i \subseteq N_{\leq 2}(v_i)$ for every i.

It remains to show that $G[N_{\leq 2}(v)]$ is 6-colorable for every $v \in V(G)$. Indeed, each $G[U_i]$ is then 6-colorable, and using disjoint sets of six colors for different U_i gives $\chi(G) \leq 6r \leq 6/\alpha$.

Fix $v \in V(G)$, and let $L_1 = N(v)$ and $L_2 = N_2(v)$; see Figure 3. We show that both $G[L_1]$ and $G[L_2]$ are 3-colorable.

Obs 1: $G[L_1]$ is 3-colorable.

Indeed, $G[L_1]$ is 2-degenerate. Otherwise, some subgraph of $G[L_1]$ would have minimum degree at least 3, and hence contain a path $x_1x_2x_3x_4$. Then $vx_1x_2x_3x_4v$ is a copy of C_5.

Obs 2: $G[L_2]$ is 3-colorable.

Let C be a nontrivial connected component of $G[L_2]$. If $xy \in E(C)$ and $a \in N(x) \cap L_1$, $b \in N(y) \cap L_1$, then $a = b$; otherwise $vaxybv$ is a copy of C_5. Thus the endpoints of every edge of C have the same unique neighbor in L_1. Since C is connected, this neighbor is the same for all vertices of C. Let $a \in L_1$

be this common neighbor. If C contained a path $x_1x_2x_3x_4$, then $ax_1x_2x_3x_4a$ would be a copy of C_5. Thus C contains no P_4, so it is 2-degenerate and hence 3-colorable. Therefore $G[L_2]$ is 3-colorable.

Finally, v has no neighbors in L_2, so it may use one of the three colors used on L_2. Hence $G[N_{\le 2}(v)]$ is 6-colorable. □

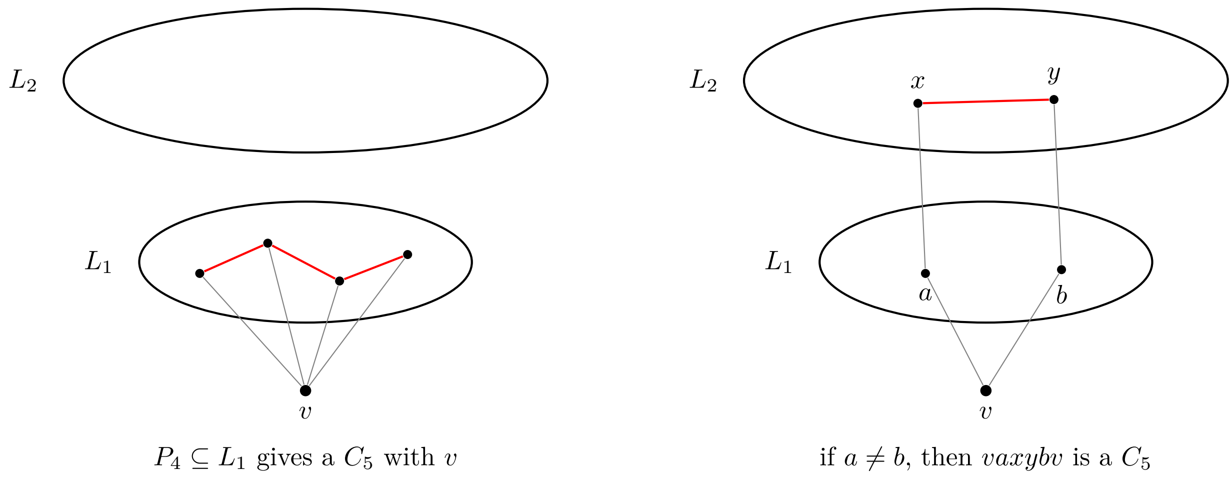

Figure 3: The sets L_1 and L_2.

2.4 The general classification

The values obtained above are part of a complete classification due to Allen et al.

Theorem 2.7 (Allen et al. [1])**.** *For every graph H with $\chi(H) = r \ge 3$, we have*

$$\delta_\chi(H) \in \left\{ \frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1} \right\}.$$

In particular, when $r = 3$,

$$\delta_\chi(H) \in \{0, 1/3, 1/2\}.$$

The graphs C_5, K_3, and $K_{2,2,2}$ realize these three values.

Proposition 2.8. $\delta_\chi(K_{2,2,2}) = 1/2$.

Proof. The upper bound follows from the Erdős–Stone theorem. Since $\chi(K_{2,2,2}) = 3$, every $K_{2,2,2}$-free graph satisfies $e(G) \le \left(\frac{1}{4} + o(1)\right) n^2$, and hence $\delta(G) \le (1/2 + o(1))n$. Therefore $\delta_\chi(K_{2,2,2}) \le 1/2$.

For the lower bound, let F be a graph of girth at least 5 and arbitrarily large chromatic number. Such graphs exist by Erdős. Take a disjoint independent set B with $|B| = |V(F)|$, and join every vertex of B to every vertex of F. Denote the resulting graph by G; see Figure 4. Then $\chi(G) \ge \chi(F)$, so $\chi(G)$ can be arbitrarily large. Also, $\delta(G) = |V(F)| = \frac{|G|}{2}$.

It remains to show that G is $K_{2,2,2}$-free. Suppose that $K_{2,2,2} \subseteq G$. Since B is independent, all vertices of the copy that lie in B must belong to the same part of the tripartition. Hence, after removing that part, the remaining four vertices lie in F and span a copy of $K_{2,2} \cong C_4$, contradicting the fact that F has girth at least 5. Therefore G is $K_{2,2,2}$-free.

Thus there exist $K_{2,2,2}$-free graphs G with $\delta(G) = |G|/2$ and arbitrarily large chromatic number, so $\delta_\chi(K_{2,2,2}) \ge 1/2$. Hence $\delta_\chi(K_{2,2,2}) = 1/2$. □

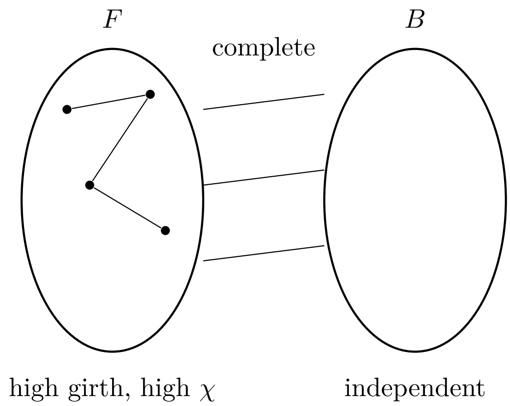

Figure 4: The lower-bound construction for $K_{2,2,2}$.

3 Homomorphism Thresholds

Bounded chromatic number is equivalent to the existence of a homomorphism to a bounded complete graph. We now strengthen the problem by requiring the bounded target graph itself to be H-free.

Definition 3.1. A map $\phi : V(G) \to V(F)$ is a *homomorphism* from G to F if $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(F)$.

Notice that $\chi(G) \le k$ if and only if there exists a homomorphism from G to K_k.

Definition 3.2. For a fixed graph H, the *homomorphism threshold* is

$$\delta_{\text{hom}}(H) := \inf\Big\{\alpha : \exists \text{ an } H\text{-free graph } F = F(H,\alpha) \text{ of bounded order such that}$$
$$\text{every } H\text{-free } G \text{ with } \delta(G) \ge \alpha|G| \text{ admits a homomorphism to } F\Big\}.$$

Since a bounded homomorphic image gives a bounded coloring, $\delta_{\text{hom}}(H) \ge \delta_\chi(H)$. For K_3, the two thresholds coincide.

Theorem 3.3 (Łuczak [9]). $\delta_{\text{hom}}(K_3) = \frac{1}{3}$.

More recently, Liu, Shangguan, Skokan and Xu proved a stronger structural statement.

Theorem 3.4 (Liu–Shangguan–Skokan–Xu [6]). *For every $\varepsilon > 0$, there exists $C = C(\varepsilon)$ such that every maximal triangle-free n-vertex graph G with $\delta(G) \ge (1/3 + \varepsilon)n$ is a blow-up of a triangle-free graph Γ with $|V(\Gamma)| \le C$.*

In particular, G admits a homomorphism to Γ, so Theorem 3.4 strengthens the upper-bound direction of Łuczak's theorem.

We sketch the proof using two ingredients. The first gives bounded VC-dimension for the relevant neighborhood hypergraph.

Lemma 3.5 (Łuczak–Thomassé [10]). *For every $\varepsilon > 0$, if G is a maximal triangle-free n-vertex graph with $\delta(G) \ge (1/3 + \varepsilon)n$, then* $\mathrm{VC}(G) = O_\varepsilon(1)$.

The second ingredient is Haussler's packing lemma.

Lemma 3.6 (Haussler packing lemma [3]). *Let $\mathcal{F} \subseteq 2^{[n]}$ have VC-dimension at most d. If $\mathcal{X} \subseteq \mathcal{F}$ satisfies $|A \triangle B| \ge s$ for every distinct $A, B \in \mathcal{X}$, then*

$$|\mathcal{X}| \le C_d \left(\frac{n}{s}\right)^d,$$

where C_d depends only on d.

Proof sketch of Theorem 3.4. Let G be a maximal triangle-free graph with $\delta(G) \geq (1/3+\varepsilon)n$. By the Łuczak–Thomassé lemma, $\mathrm{VC}(\mathcal{N}_G) = O_\varepsilon(1)$.

Fix $\rho \ll \varepsilon$, and take a maximal family $N(v_1), \dots, N(v_r)$ of pairwise ρn-separated neighborhoods. By the packing lemma, $r = O_\varepsilon(1)$. By maximality of this family, every vertex v satisfies $|N(v) \triangle N(v_i)| < \rho n$ for some i. Assigning each vertex to one such i gives a partition $V(G) = V_1 \cup \dots \cup V_r$ such that $|N(x) \triangle N(y)| < 2\rho n$ for all $x, y \in V_i$.

We show that these are the blow-up classes.

Claim 1: Each V_i is independent.

If $x, y \in V_i$ were adjacent, then triangle-freeness gives $N(x) \cap N(y) = \emptyset$, and hence $|N(x) \triangle N(y)| = d(x) + d(y) \geq 2(1/3+\varepsilon)n$, contradicting $|N(x) \triangle N(y)| < 2\rho n$.

Claim 2: For $i \neq j$, the bipartite graph $G[V_i, V_j]$ is either complete or empty.

Otherwise, choose $x, x' \in V_i$ and $y, y' \in V_j$ such that $xy \in E(G)$ but $x'y' \notin E(G)$. Since xy is an edge, x and y have no common neighbor. Thus any common neighbor of x' and y' must be a vertex whose adjacency differs between x and x', or between y and y'. Therefore $|N(x') \cap N(y')| < 4\rho n$. On the other hand, since G is maximal triangle-free and $x'y'$ is a nonedge, there exists $z \in N(x') \cap N(y')$. Triangle-freeness implies that $N(z)$ is disjoint from both $N(x')$ and $N(y')$, and hence $n \geq d(x') + d(y') + d(z) - |N(x') \cap N(y')|$. Thus $|N(x') \cap N(y')| \geq 3\varepsilon n$, a contradiction to $\rho \ll \varepsilon$.

Thus each V_i is independent and every pair (V_i, V_j) is either complete or empty. Hence G is a blow-up of a graph on $r = O_\varepsilon(1)$ vertices. □

3.1 Beyond the triangle-free case

For K_3, the chromatic and homomorphism thresholds are both $1/3$, but the two parameters need not coincide in general. For example,

$$\delta_\chi(C_5) = 0, \qquad 0 < \delta_{\mathrm{hom}}(C_5) \leq \frac{1}{5}.$$

A recent result for complete tripartite graphs gives a broader family of examples.

Theorem 3.7 (Gishboliner–Huang–Liu, 2026+)**.** *For $1 \leq s \leq t$,*

$$\delta_{\mathrm{hom}}(K_{1,s,t}) \geq \max\left\{\frac{1}{3}, \frac{s}{1+s+t}\right\}.$$

Equality holds when $3s \geq 2+2t$ and $(t-s+1)^2 \leq s$.

Remark 3.8. The homomorphism threshold exhibits several phenomena that are quite different from those of the chromatic threshold.

1. For the family $K_{1,s,s}$ with $s \geq 2$, the relevant threshold value is $\frac{s}{2s+1}$. Hence one obtains the sequence $\frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \dots$, which converges to $1/2$ as $s \to \infty$. Thus homomorphism thresholds can attain infinitely many distinct values.

2. Homomorphism thresholds need not behave monotonically with respect to the parameters of the forbidden graph. For example, $K_{1,4,4} \subseteq K_{1,4,5}$ but $\delta_{\mathrm{hom}}(K_{1,4,4}) = \frac{4}{9} > \delta_{\mathrm{hom}}(K_{1,4,5}) = \frac{4}{10}$.

We only prove the baby case $s = t = 2$, that is, $K_{1,2,2}$. In addition, we propose the following conjecture.

Conjecture 3.9. For $1 \leq s \leq t$,

$$\delta_{\mathrm{hom}}(K_{1,s,t}) = \max\left\{\frac{1}{3}, \frac{s}{1+s+t}\right\}.$$

3.2 The case $K_{1,2,2}$

We now prove the first nontrivial case of this family.

Theorem 3.10 (Baby case $s = t = 2$)**.** $\delta_{\mathrm{hom}}(K_{1,2,2}) = \frac{2}{5}$.

For the upper bound, it suffices to show that every $K_{1,2,2}$-free graph G with $\delta(G) \geq (2/5 + \varepsilon)n$ is 4-colorable. It then admits a homomorphism to K_4, which is $K_{1,2,2}$-free.

For the lower bound, one constructs $K_{1,2,2}$-free graphs G with $\delta(G) = (2/5 - o(1))|G|$ such that every bounded homomorphic image of G contains a copy of K_5, and hence a copy of $K_{1,2,2}$.

Proof of the upper bound. Choose parameters satisfying

$$\frac{1}{n} \ll \xi \ll \beta \ll \eta \ll \varepsilon.$$

We first find a partition in which both parts are almost independent.

Claim 1: There exists a partition $V(G) = A \cup B$ such that $\Delta(G[A]), \Delta(G[B]) \leq 2\eta n$.

Since G is $K_{1,2,2}$-free, for every vertex v, $G[N(v)]$ is $K_{2,2}$-free. By the Kővári–Sós–Turán theorem [5], $e(G[N(v)]) = O(n^{3/2})$. Each triangle contributes one edge to $G[N(v)]$ for each of its three vertices, so

$$3\#K_3(G) = \sum_{v \in V(G)} e(G[N(v)]) = O(n^{5/2}) \leq \xi n^3.$$

By the Triangle Removal Lemma, we may delete at most βn^2 edges to obtain a triangle-free spanning subgraph G'.

Let Z be the set of vertices incident with at least ηn deleted edges. Since at most βn^2 edges were deleted, $|Z| \leq \frac{2\beta}{\eta} n \ll \eta n$. Every vertex outside Z loses fewer than ηn edges, so

$$\delta(G' - Z) \geq \delta(G) - \eta n - |Z| > \frac{2}{5}|V(G' - Z)|.$$

By Theorem 1.4, $G' - Z$ is bipartite; let (A_0, B_0) be its bipartition. Since G' has no edges inside A_0 or B_0, $\Delta(G[A_0]), \Delta(G[B_0]) \leq \eta n$.

It remains to place the vertices of Z. We claim that every $z \in Z$ has fewer than ηn neighbors in at least one of A_0 and B_0. Otherwise, for some $z \in Z$, let

$$X = N(z) \cap A_0, \qquad Y = N(z) \cap B_0,$$

where $|X|, |Y| \geq \eta n$. Since G is $K_{1,2,2}$-free, $G[X, Y]$ is $K_{2,2}$-free, so the Kővári–Sós–Turán theorem gives $e(X, Y) = O(n^{3/2})$. On the other hand, for every $x \in X$ and $y \in Y$,

$$d(x, Y) \geq \delta(G) - |B_0 \setminus Y| - \eta n - |Z|,$$

and similarly $d(y, X) \geq \delta(G) - |A_0 \setminus X| - \eta n - |Z|$. Moreover, $|X| + |Y| \geq \delta(G) - |Z|$. Since $|A_0| + |B_0| = n - |Z|$, we obtain $d(x, Y) + d(y, X) \geq 3\delta(G) - n - 2\eta n - 2|Z| > \frac{n}{5}$. Thus either every $x \in X$ has at least $n/10$ neighbors in Y, or every $y \in Y$ has at least $n/10$ neighbors in X. Since $|X|, |Y| \geq \eta n$, this gives $e(X, Y) \geq \frac{\eta}{10} n^2$, contradicting $e(X, Y) = O(n^{3/2})$ for sufficiently large n.

Therefore every $z \in Z$ has fewer than ηn neighbors in at least one of A_0 and B_0. Place each z on such a side. Since $|Z| \ll \eta n$, the resulting partition $V(G) = A \cup B$ satisfies $\Delta(G[A]), \Delta(G[B]) \leq 2\eta n$, proving Claim 1.

Claim 2: $\Delta(G[A]), \Delta(G[B]) \leq 1$.

Suppose that some $x \in A$ has two distinct neighbors $v_1, v_2 \in A$. By Claim 1, every vertex has at most $2\eta n$ neighbors on its own side. Hence

$$d_B(u) \geq \left(\frac{2}{5} + \varepsilon - 2\eta\right) n \qquad \text{for every } u \in A,$$

and similarly $d_A(u) \geq \left(\frac{2}{5}+\varepsilon-2\eta\right)n$ for every $u \in B$. Consequently,

$$|A|,|B| \in \left[\left(\frac{2}{5}+\varepsilon-2\eta\right)n,\ \left(\frac{3}{5}-\varepsilon+2\eta\right)n\right].$$

Applying inclusion–exclusion to $N_B(x), N_B(v_1), N_B(v_2)$ gives

$$\begin{aligned}|N_B(x)\cap N_B(v_1)\cap N_B(v_2)| &\geq d_B(x)+d_B(v_1)+d_B(v_2)-2|B| \\ &\geq (5\varepsilon-10\eta)n.\end{aligned}$$

Since $\eta \ll \varepsilon$, this set contains two distinct vertices y_1, y_2. Then $\{x\}$, $\{v_1, v_2\}$ and $\{y_1, y_2\}$ form the three parts of a copy of $K_{1,2,2}$, a contradiction. Thus $\Delta(G[A]) \leq 1$, and similarly $\Delta(G[B]) \leq 1$.

Hence both $G[A]$ and $G[B]$ are a matching together with isolated vertices, and so each is 2-colorable. Therefore $\chi(G) \leq 4$, giving $\delta_{\text{hom}}(K_{1,2,2}) \leq \frac{2}{5}$. □

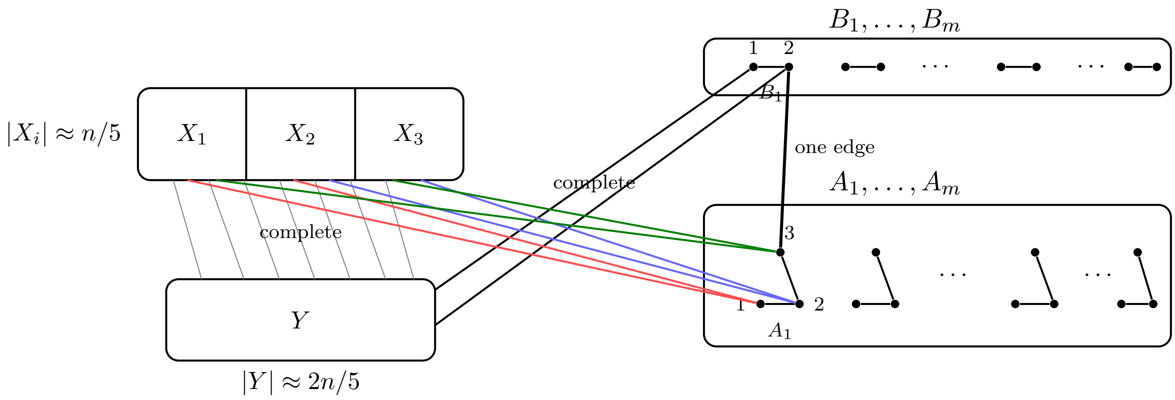

Figure 5: The lower-bound construction for $K_{1,2,2}$.

Proof of the lower bound. Fix $C \in \mathbb{N}$, and choose parameters such that

$$\frac{1}{n} \ll \frac{1}{m} \ll \beta \ll \frac{1}{C}.$$

We first construct a pseudorandom template. For each $i, j \in [m]$, let $B_i = \{b_i^1, b_i^2\} \cong K_2$ and $A_j = \{a_j^1, a_j^2, a_j^3\} \cong K_3$. For every pair (B_i, A_j), choose independently and uniformly a label $(p,q) \in [2]\times[3]$, corresponding to the edge $b_i^p a_j^q$.

($\star$) For every $I, J \subseteq [m]$ with $|I|, |J| \geq \beta m$, and every $(p,q) \in [2]\times[3]$, some pair $(i,j) \in I \times J$ receives the label (p,q).

Such a template exists. Indeed, for fixed I, J and (p,q), the probability that (p,q) does not occur on $I \times J$ is at most

$$\left(\frac{5}{6}\right)^{\beta^2 m^2} \leq \exp\left(-\frac{\beta^2 m^2}{6}\right).$$

There are at most 4^m choices for (I,J) and six labels, so the probability that ($\star$) fails is at most $6 \cdot 4^m \exp(-\beta^2 m^2/6) = o(1)$. Thus we may fix a template satisfying ($\star$).

We now construct G; see Figure 5. Set $N := n - 5m$, and assume for simplicity that $5 \mid N$. Add four disjoint independent sets X_1, X_2, X_3, Y with $|X_1| = |X_2| = |X_3| = N/5$ and $|Y| = 2N/5$. Join Y completely to $X_1 \cup X_2 \cup X_3$ and to $B_1 \cup \cdots \cup B_m$. For every $j \in [m]$ and $q \in [3]$, join a_j^q completely to $X_q \cup X_{q+1}$, where the indices are taken modulo 3. No further edges are added.

We first check that G is $K_{1,2,2}$-free. It is enough to show that $G[N(v)]$ is C_4-free for every $v \in V(G)$. If $v \in Y$ or $v \in X_i$, then $G[N(v)]$ is a matching together with isolated vertices. If $v \in B_i$, then $G[N(v)]$ is a star together with isolated vertices, while if $v \in A_j$, it is a tree together with isolated vertices. Thus $G[N(v)]$ contains no C_4.

The minimum degree follows directly from the construction. For $x \in X_i$, we have $d(x) = |Y| + 2m = 2n/5$. Every vertex in a block B_i or A_j has at least $2N/5$ neighbors, while every vertex of Y has degree $3N/5 + 2m$. Hence

$$\delta(G) \geq \frac{2N}{5} = \frac{2n}{5} - 2m = \left(\frac{2}{5} - o(1)\right) n.$$

It remains to rule out bounded $K_{1,2,2}$-free homomorphic images. Let ϕ be a homomorphism from G to a graph Γ with $|V(\Gamma)| \leq C$. By the pigeonhole principle, there exist $x_1, x_2 \in V(\Gamma)$ and $I \subseteq [m]$ with $|I| \geq m/C^2 \geq \beta m$ such that $\phi(b_i^1) = x_1$ and $\phi(b_i^2) = x_2$ for every $i \in I$. Similarly, there exist $y_1, y_2, y_3 \in V(\Gamma)$ and $J \subseteq [m]$ with $|J| \geq m/C^3 \geq \beta m$ such that $\phi(a_j^q) = y_q$ for every $j \in J$ and $q \in [3]$.

The internal edges of the blocks imply that $x_1x_2 \in E(\Gamma)$ and $\{y_1, y_2, y_3\}$ spans a triangle. By property $(\star)$, every label $(p, q) \in [2] \times [3]$ occurs on some pair $(i, j) \in I \times J$, so $x_p y_q \in E(\Gamma)$. Thus x_1, x_2, y_1, y_2, y_3 span a copy of K_5. Since K_5 contains $K_{1,2,2}$, no $K_{1,2,2}$-free graph on at most C vertices can be a homomorphic image of G. As C was arbitrary, $\delta_{\mathrm{hom}}(K_{1,2,2}) \geq 2/5$. $\square$

4 Additive Analogues: Homogeneous Linear Equations

Consider a homogeneous linear equation

$$\mathcal{L}: \qquad \sum_{i=1}^{k} c_i x_i = 0, \qquad k \geq 3.$$

A set $A \subseteq \mathbb{F}_p$ is *$\mathcal{L}$-solution-free* if it contains no solution of the prescribed type. In [7], the coordinates of a solution are required to be pairwise distinct.

The additive analogue of the Turán problem asks how large an $\mathcal{L}$-solution-free set can be. To introduce a chromatic analogue, consider the Cayley digraph $\mathrm{Cay}(\mathbb{F}_p, A)$, where $u \to v$ if $v - u \in A$. Every vertex has out-degree $|A|$, so the density $|A|/p$ plays the role of the minimum-degree density.

Definition 4.1 (Chromatic threshold of a linear equation [7]). The *chromatic threshold* of $\mathcal{L}$ is

$$\delta_\chi(\mathcal{L}) := \inf\Big\{d > 0 : \ \exists C = C(\mathcal{L}, d) \text{ such that for every prime } p \text{ and every } \mathcal{L}\text{-solution-free } A \subseteq \mathbb{F}_p,$$
$$|A| \geq dp \Longrightarrow \chi(\mathrm{Cay}(\mathbb{F}_p, A)) \leq C\Big\}.$$

The equation $\mathcal{L}$ is *translation-invariant* if $\sum_{i=1}^{k} c_i = 0$. Indeed, in this case translating every variable x_i to $x_i + t$ preserves the equation.

The basic example is the three-term arithmetic progression equation $x - 2y + z = 0$. More generally, the following result characterizes when every solution-free set has zero density.

Theorem 4.2 (Roth-type density criterion, as stated in [7]). *Let $\mathcal{L} : \sum_{i=1}^{k} c_i x_i = 0$ with $k \geq 3$. Every $\mathcal{L}$-solution-free set $A \subseteq \mathbb{F}_p$ has size $o(p)$ as $p \to \infty$ if and only if*

$$\sum_{i=1}^{k} c_i = 0.$$

For the chromatic threshold, the corresponding condition is weaker.

Theorem 4.3 (Liu–Wu–Yang–Zhang, 2026 [7]). *Let $\mathcal{L} : \sum_{i=1}^{k} c_i x_i = 0$ with $k \geq 3$. Then*

$$\delta_\chi(\mathcal{L}) = 0 \quad \Longleftrightarrow \quad \exists I \subseteq [k],\ |I| \geq 3,\ \sum_{i \in I} c_i = 0.$$

Thus $\delta_\chi(\mathcal{L}) = 0$ does not require $\mathcal{L}$ itself to be translation-invariant. For example, the equation $x + y + z = 2w$ has coefficient vector $(1, 1, 1, -2)$, whose total sum is nonzero. However, the subcollection $(1, 1, -2)$ has sum zero, so Theorem 4.3 gives $\delta_\chi(x + y + z = 2w) = 0$.

Hence the condition for zero chromatic threshold sits between translation invariance and the existence of an arbitrary nonempty zero-sum subcollection:

$$\sum_{i=1}^{k} c_i = 0 \quad \Longrightarrow \quad \exists I \subseteq [k],\ |I| \geq 3,\ \sum_{i \in I} c_i = 0 \quad \Longrightarrow \quad \exists \emptyset \neq I \subseteq [k],\ \sum_{i \in I} c_i = 0.$$

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