Applications of Combinatorial Nullstellensatz

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Contents

Chapter 1

Combinatorial Nullstellensatz

1.1 Non-zero points of multi-variant polynomials

Assume $\mathbb{F}$ is a field, and $f(x_1, x_2, \dots, x_n) \in \mathbb{F}[x_1, x_2, \dots, x_n]$ is a polynomial with coefficients in $\mathbb{F}$. If $\mathbf{a} := (a_1, a_2, \dots, a_n) \in \mathbb{F}^n$ and $f(\mathbf{a}) \neq 0$, then $\mathbf{a}$ is called a *non-zero point* of f. For subsets $S_1, S_2, \dots, S_n$ of $\mathbb{F}$, the set

$$\mathbf{S} := S_1 \times S_2 \times \cdots \times S_n = \{(a_1, a_2, \dots, a_n) : a_i \in S_i\} \subseteq \mathbb{F}^n$$

is called *an n-dimensional grid* over $\mathbb{F}$. The sets S_i are usually finite subsets of $\mathbb{F}$.

There are many combinatorial problems for which the existence of a solution is equivalent to the existence of non-zero points of a polynomial in a given grid. Below are three examples of such problems in graph theory.

The first example is graph coloring, which is a central topic of this lecture note.

For a graph G, we denote by $V(G)$ and $E(G)$ its vertex set and edge set. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the natural number set, and we denote by $\mathbb{N}^G$ the set of mappings $g : V(G) \to \mathbb{N}$.

Definition 1.1.1 *Assume G is a graph. A* proper coloring *of G is a mapping $\phi : V(G) \to C$ such that $\phi(u) \neq \phi(v)$ for each edge $uv \in E(G)$, where C is a subset of a field, and is called the set of colors.*

Assume G is a graph. An *orientation* of G is a digraph D obtained from G by replacing each edge xy with an arc (x, y) or (y, x).

Example 1.1.2 *Assume G is a graph. Let $<$ be an ordering of the vertices of G. The graph polynomial P_G with variables $\{x_v : v \in V(G)\}$ defined as*

$$P_G(x_v : v \in V(G)) = \prod_{uv \in E(G), u<v} (x_u - x_v).$$

Sometimes we may define the polynomial $P_G(x_v : v \in V(G))$ through an orientation D as follows:

$$P_G(x_v : v \in V(G)) = \prod_{(u,v) \in E(D)} (x_u - x_v).$$

Using the ordering amounts to choose the orientation D in which an edge uv is oriented as (u, v) when $u < v$.

Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real numbers and the field of complex numbers, respectively. Then $P_G(x_v : v \in V(G)) \in \mathbb{R}[x_v : v \in V(G)] \subseteq \mathbb{C}[x_v : v \in V(G)]$.

For a mapping $\phi : V \to \mathbb{R}$, let $P_G(\phi) = P_G(\phi(v) : v \in V)$ be the evalutation of P_G at $x_v = \phi(v)$ for $v \in V(G)$. Then ϕ is a proper coloring of G if and only if $P_G(\phi) \neq 0$, i.e., $(\phi(v) : v \in V)$ is a non-zero point of the polynomial P_G.

In graph coloring, a central problem is whether G has a proper coloring ϕ with colors $\phi(v)$ chosen from given sets, which then is equivalent to the problem whether the graph polynomial P_G has a non-zero point in a given grid.

Definition 1.1.3 *A graph G is called* k-colorable *if G has a proper coloring ϕ using k colors, i.e., $|\phi(V(G))| = k$. The* chromatic number *of G is*

$$\chi(G) = \min\{k : G \textit{ is } k\textit{-colorable}\}.$$

Thus the problem whether G is k-colorable is equivalent to the problem whether the polynomial P_G has a non-zero point in the grid $|S|^{V(G)}$ for a k-subset S of $\mathbb{R}$.

Definition 1.1.4 *A* list assignment *of G is a mapping L that assigns to each vertex v of G a set $L(v) \subseteq \mathbb{R}$ as permissible colors for v. We say G is* L-colorable *if G has a proper coloring ϕ with $\phi(v) \in L(v)$ for each vertex v.*

Definition 1.1.5 *For $f \in \mathbb{N}^G$, an* f-list assignment *is a list assignment L with $|L(v)| \geq f(v)$ for each vertex v. We say G is* f-choosable *if G is L-colorable for any f-list assignment L of G. If $f(v) = k$ for all vertices v, then f-choosable is called* k-choosable. *The* choice number *of G is defined as*

$$\mathrm{ch}(G) = \min\{k : G \textit{ is } k\textit{-choosable.}\}$$

Assume $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then G is f-choosable if and only if any grid $L(v_1) \times L(v_2) \times \ldots \times L(v_n)$ with $|L(v_i)| \geq f(v_i)$ contains a non-zero point of P_G.

The second example is graph orientation with restrictions on the out-degree of vertices.

Definition 1.1.6 *Given an orientation D of G, and a vertex v of G, $E_D^+(v) = \{(v,u) : (v,u) \in E(D)\}$ is the set of* out-going edges *of v, and $E_D^-(v) = \{(u,v) : (u,v) \in E(D)\}$ is the set of* in-coming edges *of v. The cardinality of $E_D^+(v)$ and $E_D^-(v)$ are the* out-degree *and* in-degree *of v, and are denoted by $d_D^+(v)$ and $d_D^-(v)$, respectively.*

Definition 1.1.7 *Assume G is a graph and $F : V(G) \to 2^{\mathbb{N}}$ is a list assignment, which assigns to each vertex v a set $F(v)$ of integers. An orientation D of G is called* F-avoiding *if for each vertex v, $d_D^+(v) \notin F(v)$.*

Given such a list assignment F, the question is whether G has an F-avoiding orientation. The following conjecture was proposed by Akbari, Dalirrooyfard, Ehsani, Ozeki and Sherkati [1], and remains open.

Conjecture 1.1.8 *Assume G is a graph and F is a list assignment that assigns to each vertex v a set $F(v)$ of integers. If $|F(v)| \leq (d_G(v) - 1)/2$, then G has an F-avoiding orientation.*

Given an orientation D of G, the *imbalance* of a vertex $v \in V(G)$ is the difference $d_D^+(v) - d_D^-(v)$ between out-degree and in-degree of v. As

$$d_D^+(v) = \frac{1}{2}\left(d(v) + \left(d_D^+(v) - d_D^-(v)\right)\right),$$

$d_D^+(v) \notin F(v)$ if and only if $d_D^+(v) - d_D^-(v) \notin F'(v)$, where

$$F'(v) \colon = 2F(v) - d(v).$$

We say an orientation D of G is *imbalance F'-avoiding* if $d_D^+(v) - d_D^-(v) \notin F'(v)$ for each vertex v. Therefore, G has an F-avoiding orientation if and only if G has an imbalance F'-avoiding orientation. Conjecture 1.1.8 is equivalent to the following conjecture.

Conjecture 1.1.9 *Assume G is a graph and F is a list assignment that assigns to each vertex v a set $F(v)$ of integers. If $|F(v)| \leq (d_G(v) - 1)/2$, then G has an imbalance F-avoiding orientation.*

Assume G has a preset orientation D. Then any mapping $\phi : E(G) \to \{1, -1\}$ defines an orientation D_ϕ defined as follows: If $\phi(e) = 1$, then D_ϕ and D agree on e, i.e., D and D_ϕ orient e in the same direction. If $\phi(e) = -1$, then D_ϕ and D disagree on e, i.e., D and D_ϕ orient e in opposite directions. For each edge e and vertex v of G, let

$$a_{ev} = \begin{cases} 1, & e \in E_D^+(v), \\ 0, & e \notin E_D(v), \\ -1, & e_j \in E_D^-(v). \end{cases}$$

It can be easily checked that for a mapping $\phi : E(G) \to \{1, -1\}$,

$$a_{ev}\phi(e) = \begin{cases} 1, & e \in E_{D_\phi}^+(v), \\ 0, & e \notin E_{D_\phi}(v), \\ -1, & e_j \in E_{D_\phi}^-(v). \end{cases}$$

Therefore

$$d_{D_\phi}^+(v) - d_{D_\phi}^-(v) = \sum_{e \in E(G)} a_{ev}\phi(e).$$

Now we associate to each edge e of G a variable x_e. For any list assignment F' of G, let $g_{D,F'}(x_e : e \in E(G))$ be the polynomial defined as

$$g_{D,F'}(x_e : e \in E(G)) = \prod_{v \in V(G)} \prod_{a \in F'(v)} (\sum_{e \in E(G)} (a_{ev}x_e - a).$$

For a mapping $\phi : E(G) \to \{1, -1\}$, $g_{D,F'}(\phi) \neq 0$ implies that $d_{D_\phi}^+(v_i) - d_{D_\phi}^-(v) \notin F'(v)$ for each vertex v. Then G has an imbalance F'-avoiding orientation if and only if the grid $\{-1, 1\}^{|E(G)|}$ contains a non-zero point of $g_{D,F'}$.

The third example is about edge-weighting.

Definition 1.1.10 *An* edge-weighting *of a graph G is a mapping $\phi : E(G) \to \mathbb{R}$. Given an edge-weighting ϕ, the* vertex-sum *$S_\phi : V(G) \to \mathbb{R}$ is defined as*

$$S_\phi(v) := \sum_{e \in E(v)} \phi(e).$$

We say ϕ is a proper edge-weighting *if S_ϕ is a proper vertex-coloring of G, i.e., for any edge $e = uv$ of G, $S_\phi(u) \neq S_\phi(v)$.*

Definition 1.1.11 *Given a positive integer k, we say G has a proper k-*edge weighting *if there is a proper edge-weighting ϕ of G such that $\phi(e) \in \{1, 2, \ldots, k\}$ for every* *edge e. We say a graph G is* k-edge weight choosable *if for any list assignment L that assigns to each edge e of G a set $L(e)$ of k real numbers, there is a proper edge-weighting ϕ with $\phi(e) \in L(e)$ for each edge e.*

If G has an isolated edge $e = uv$, then for any edge-weighting ϕ of G, $S_\phi(u) = S_\phi(v) = \phi(e)$. So G has no proper k-edge weighting for any k. It is known [19] that every graph without isolated edge has a proper 3-edge weighting. For list edge-weighting, the following conjecture was proposed by Bartnicki, Grytczuk and Niwczyk [3], and remains open.

Conjecture 1.1.12 *Every graph without isolated edge is* 3*-edge weight choosable.*

For a graph G, we associate to each edge $e \in E(G)$ a variable x_e. Let D be an orientation of G and let h_D be the polynomial with variables $\{x_e : e \in E(G)\}$ defined as

$$h_D(x_e : e \in E) := \prod_{uv \in E(D)} \Big(\sum_{e \in E(u)} x_e - \sum_{e \in E(v)} x_e\Big).$$

It is easy to see that for a mapping $\phi : E(G) \to \mathbb{R}$, $h_D(\phi) \neq 0$ if and only if ϕ is a proper edge weighting of G. Therefore, to find a proper edge-weighting of G is equivalent to find a non-zero point of h_D. Similarly for the graph polynomial, the orientation D is just used as a reference. If D' is another orientation of G, then either $h_D = h_{D'}$ or $h_D = -h_{D'}$. A point $\mathbf{x} \in \mathbb{R}^{E(G)}$ is a non-zero point of h_D if and only if it is a non-zero point of $h_{D'}$. We may write h_G for h_D (with D be an arbitrary orientation of G). Conjecture 1.1.12 is equivalent to say that any grid $\prod_{e \in E(G)} L(e)$ with $|L(e)| \geq 3$ contains a non-zero point of h_G.

1.2 Combinatorial Nullstellensatz

We have shown three examples of problems in graph theory that are equivalent to find a non-zero point in a given grid for a certian polynomial.

Combinatorial Nullstellensats is a theorem that gives a sufficient condition for a existence of a non-zero point of a multi-variable polynomial in a given grid.

Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a sequence of non-negative integers. We denote by $\mathbf{x}^\mathbf{t}$ the monomial $\prod_{i=1}^n x_i^{t_i}$. Every polynomial is a linear combination of monomials. For $f(x_1, x_2, \ldots, x_n) \in \mathbb{F}[x_2, x_2, \ldots, x_n]$, we denote by $c_{f,\mathbf{t}}$ the coefficient of $\mathbf{x}^\mathbf{t}$ in f. We say $\mathbf{x}^{\mathbf{t}}$ is a *non-vanishing monomial* of f if $c_{f,\mathbf{t}} \neq 0$. We denote by $\mathrm{Mon}(f)$ the set of non-vanishing monomials of f. Thus

$$f(x_1, x_2, \ldots, x_n) = \sum_{\mathbf{x}^{\mathbf{t}} \in \mathrm{Mon}(f)} c_{f,\mathbf{t}} \mathbf{x}^{\mathbf{t}}.$$

The *degree* of $\mathbf{x}^{\mathbf{t}}$ is $\deg(\mathbf{x}^{\mathbf{t}}) = \sum_{i=1}^{n} t_i$. The degree of f, denoted by $\deg(f)$, is the highest degree of its non-vanishing monomials.

Theorem 1.2.1 *[Combinatorial Nullstellensatz (CNS) [2]] Let $\mathbb{F}$ be a field and $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a polynomial of degree at least 1. Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a sequence of non-negative integers. Suppose that $\mathbf{x}^{\mathbf{t}}$ is a highest degree non-vanishing monomial of f. Then any n-dimensional grid $\mathbf{S} = S_1 \times S_2 \times \ldots \times S_n$ of $\mathbb{F}$ with $|S_i| \geq t_i + 1$ contains a non-zero point of f.*

Proof. First we consider the case that for each variable x_i, the highest degree of x_i in f is at most t_i, and prove the following lemma.

Lemma 1.2.2 *Assume $f(x_1, x_2, \ldots, x_n) \in F[x_1, x_2, \ldots, x_n]$, the highest degree of x_i in f is at most t_i, and $\mathbf{S} = S_1 \times \ldots \times S_n$, where $|S_i| = t_i + 1$ for $i = 1, 2, \ldots, n$. If f vanishes on $\mathbf{S}$ (i.e., $f(\mathbf{a}) = 0$ for each $\mathbf{a} \in \mathbf{S}$), then $f \equiv 0$.*

Proof. If $n = 1$, then the result is equivalent to say that a non-zero polynomial $f(x)$ of degree at most t has at most t distinct roots.

Assume $n \geq 2$. For each tuple $(x_1, \ldots, x_{n-1}) \in S_1 \times \ldots S_{n-1}$,

$$g(x_n) = f(x_1, \ldots, x_n) = \sum_{j=0}^{t_n} f_j(x_1, \ldots, x_{n-1}) x_n^j$$

is a polynomial of x_n of degree at most t_n. As $g(x_n)$ vanishes on a set S_n of $t_n + 1$ elements, $g \equiv 0$. I.e., for $j = 0, 1, \ldots, t_n$, $f_j(x_1, \ldots, x_{n-1})$ vanishes on $S_1 \times \ldots S_{n-1}$. As the degree of x_i in $f_j(x_1, \ldots, x_{n-1})$ is at most t_i, by induction hypothesis, $f_j \equiv 0$. Therefore $f \equiv 0$. ∎

Now we prove Theorem 1.2.1.

If each variable x_i has degree at most t_i in f, then since $f \not\equiv 0$ (as $\mathbf{x}^{\mathbf{t}}$ is a non-vanishing monomial of f), it follows from Lemma 1.2.2 that f has a non-zero point in $\mathbf{S}$.

Assume for some i, f has a non-vanishing monomial $\prod_{j=1}^n x_j^{d_j}$ in which the degree of x_i is at least t_i+1. Let

$$g_i(x_i) = \prod_{c \in S_i}(x_i - c) = x_i^{t_i+1} - h_i(x_i),$$

where $h_i(x_i)$ is of degree at most t_i. Replace $x_i^{t_i+1}$ in the monomial $\prod_{j=1}^n x_j^{d_j}$ by $x_i^{t_i+1} - g_i(x_i) = h_i(x_i)$. This reduces the degree of x_{i+1} by 1 in this monomial. Repeat this process, until we obtain a polynomial $\tilde{f}$ in which the degree of x_i is at most t_i for each i. Observe that the coefficient of the monomial $\mathbf{x}^{\mathbf{t}} = \prod_{i=1}^n x_i^{t_i}$ in $\tilde{f}$ is the same as in f, and hence is non-zero. It follows from Lemma 1.2.2 that $\tilde{f}$ has a non-zero point in $\mathbf{S}$. As $g_i(x_i) = 0$ for all $x_i \in S_i$, we conclude that $f(\mathbf{x}) = \tilde{f}(\mathbf{x})$ for $\mathbf{x} \in \mathbf{S}$. Thus f has a non-zero point in $\mathbf{S}$. This completes the proof of Theorem 1.2.1. ∎

For a graph G and $g \in \mathbb{N}^G$, we denote by $c_{G,g}$ the coefficient of the monomial $\mathbf{x}^g = \prod_{v \in V(G)} x_v^{g(v)}$ in the expansion of P_G.

Note that the graph polynomial P_G is a homogeneous polynomial (i.e., all its non-vanishing monomials have the same degree). The CNS is particularly suitable for the study of graph list coloring, as shown by Theorem 1.2.3, which is an easy consequence of the CNS.

Theorem 1.2.3 *Assume G is a graph, and $f, g \in \mathbb{N}^G$. If $c_{G,g} \neq 0$ and $f(v) \geq g(v)+1$ for each vertex v, then G is f-choosable.*

The existence of an imbalance F'-avoiding orientation of a graph G is equivalent to the existence of non-zero point in the polynomial $g_{D,F'}$ (for an orientation D of G). By applying CNS, we have the following corollary.

Corollary 1.2.4 *Assume $q : V(G) \to \mathbb{N}$ and F' is a list assignment that assigns to each vertex v a set $F'(v)$ of $q(v)$ integers. If $g_{D,F'}$ has a highest degree non-vanishing monomial $\prod_{e \in E'} x_e$ for some subset E' of $E(G)$, then G has an imbalance F'-avoiding orientation D.*

Let

$$g_{D,q}(x_e : e \in E(D)) = \prod_{v \in V(G)} \sum_{e \in E(D)} (a_{ev}x_e)^{q(v)}.$$

Then $g_{D,F'}$ and $g_{D,q}$ have the same set of highest non-vanishing monomials. Therefore, we have the following corollary.

Corollary 1.2.5 *Assume $q : V(G) \to \mathbb{N}$ and $g_{D,q}$ has a highest degree non-vanishing monomial $\prod_{e \in E'} x_e$ for some subset E' of $E(G)$. Then for any list assignment F' of G with $|F'(v)| \leq q(v)$, G has an imbalance F'-avoiding orientation. Consequently, for any list assignment F of G with $|F(v)| \leq q(v)$, G has an F-avoiding orientation.*

A proper edge weighting is equivalent to a non-zero point of the polynomial h_G. By using CNS, we have the following corollary.

Corollary 1.2.6 *If h_G has a non-vanishing monomial $\prod_{e \in E} x_e^{t(e)}$ with $t(e) \leq k-1$ for each edge e, then G is edge-weight k-choosable.*

We shall apply CNS to show in Chapter ? that every graph G with no isolated edge is edge-weight 5-choosable. For $k = 3, 4$, the question whether every graph with no isolated edge is edge-weight k-choosable remain open.

Chapter 2

Graph polynomial

To apply CNS for a combinatorial problem, we first construct a polynomial whose non-zero points correspond to solutions of the problem. Then we need to verify the conditions of CNS are satisfied that ensures the existence of a non-zero point in a given grid. We have already constructed the plynomials for the graph colouring problem, the list out-degree avoiding orientation problem and the edge-weighting problem. We shall focus on these problems in this lecture note. The task is to show that the condition of CNS is satisfied for the polynomials in concern. This amounts to show that certain monomials are non-vanishing for a given polynomial. In this chapter, we consider the graph polynomial and show how this is done for some families of graphs.

2.1 Graph polynomial and Alon-Tarsi number

Given a graph G, the graph polynomial $P_G(x_v : v \in V(G))$ is defined as

$$P_G(x_v : v \in V(G)) = \prod_{uv \in E(D)} (x_u - x_v),$$

where D is an orientation of G. To indicate which orientation D is used in the definition, we may write P_G as P_D. For different orientations D and D' of G, we have $P_D = (-1)^q P_{D'}$, where q is the number of edges of G that are oriented differently in D and D'. Usually, we choose the orientation D by ordering the vertices of G and orient an edge uv as (u, v) if $u < v$. The polynomial P_G is defined as

$$P_G(x_v : v \in V(G)) = \prod_{uv \in E(G), u<v} (x_u - x_v).$$

Definition 2.1.1 *For $g \in \mathbb{N}^G$, $c_{G,g}$ denotes the coefficient of $\mathbf{x}^g = \prod_{v \in V(G)} x_v^{g(v)}$ in the expansion of $P_G(\mathbf{x})$. We say $\mathbf{x}^g$ is a non-vanishing monomial of P_G if $c_{G,g} \neq 0$.*

For a graph G, let

$$\mathrm{mon}(P_G) = \{\mathbf{x}^g : c_{G,g} \neq 0\}$$

be the set of non-vanishing monomials of P_G. Note that although the definition of P_G depends on the orientation D of G, $\mathrm{mon}(P_G)$ is independent of the choice of orientation D.

Definition 2.1.2 *Assume that G is a graph and $f \in \mathbb{N}_0^G$. We say that G is f-*Alon–Tarsi *(f-AT for short), if P_G has a non-vanishing monomial $\mathbf{x}^g$ with $g(v) \leq f(v) - 1$ for $v \in V(G)$. We say G is k-AT if G is f-AT for the constant function $f(v) = k$ for every $v \in V(G)$. The* Alon-Tarsi number *$AT(G)$ of G is defined as*

$$AT(G) = \min\{k : \ G \textit{ is } k\textit{-AT} \}.$$

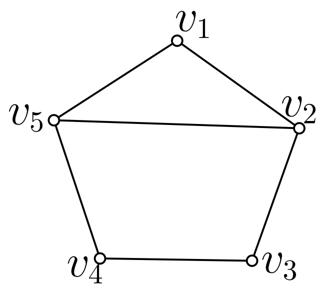

Figure 2.1: The grah G

For example, for the graph G in Figure 2.1,

$$\begin{aligned} P_G(x_1, x_2, \ldots, x_5) &= (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)(x_1 - x_5)(x_5 - x_2) \\ &= 2x_1^2 x_2 x_3 x_4 x_5 + \cdots \end{aligned}$$

and $AT(G) = 3$, as $x_1^2 x_2 x_3 x_4 x_5$ is a non-vanishing monomial of $P_G(\mathbf{x})$, and the polynomial is of degree 6 and hence in any non-vanishing monomial, at least one of the variables has exponent at least 2.

By Theorem 1.2.3, if G is f-AT, then G is f-choosable, and hence

$$ch(G) \leq AT(G).$$

We shall derive upper bounds for $AT(G)$ for some class of graphs. This gives upper bounds for the choice number $ch(G)$ of G. For those graph classes, the upper bounds for $ch(G)$ are usually quite tight. Nevertheless, for general graphs, the difference $AT(G) - ch(G)$ can be arbitrarily large.

We first give example of graphs G for which the difference $AT(G) - ch(G)$ can be arbitrarily large.

Definition 2.1.3 *The* maximum average density *of a graph G is defined as*

$$\mathrm{mad}(G) = \max\{\frac{|E(H)|}{|V(H)|} : H \subseteq G\}.$$

Theorem 2.1.4 *For every graph G,*

$$AT(G) \geq \lceil \mathrm{mad}(G) \rceil + 1.$$

Proof. Assume $\mathrm{mad}(G) = \frac{|E(H)|}{|V(H)|}$. Then for any non-vanishing monomial $\mathbf{x}^g$ of f_G, $\sum_{v \in V(H)} g(v) \geq |E(H)|$, and hence there is a vertex $v \in V(H)$ with $g(v) \geq |E(H)|/|V(H)| = \mathrm{mad}(G)$. Therefore $AT(G) \geq \lceil \mathrm{mad}(G) \rceil + 1$. ∎

It follows from Theorem 2.1.4 that $AT(K_{n,n}) \geq \frac{n}{2} + 1$. On the other hand, the following proposition shows that $ch(K_{n,n}) \leq \log_2 n + 2$. Hence the difference $AT(G) - ch(G)$ can be arbitrarily large.

Theorem 2.1.5 *For any positive integer n, $ch(K_{n,n}) \leq \log_2 n + 2$.*

Proof. Let $k = \lfloor \log_2 n \rfloor + 2$ and L is a k-list assignment of G. Without loss of generality, assume that $\bigcup_{v \in V(G)} L(v) = \{1, 2, \ldots, q\}$. Let $X_i = \{v : i \in L(v)\}$. Assume that the two partite sets of G are A, B. We assign weight $w(v) = 1$ to each vertex v of G and for each subset X of $V(G)$, let $w(X) = \sum_{v \in X} w(v)$ be the weight of X.

We color the vertices of G in q steps,

The weights of vertices in G will be updated in each step, and we always use $w(v)$ to denote the current weight of v. Let U_i be the set of uncolored vertices before the ith step. So $U_1 = V(G)$. In the ith step, if $w(X_i \cap U_i \cap A) \geq w(X_i \cap U_i \cap B)$, then we color vertices in $X_i \cap U_i \cap A$ with color i and double the weight of each vertex in $X_i \cap U_i \cap B$. Otherwise, we color vertices in $X_i \cap U_i \cap B$ with color i and double the weight of each vertex in $X_i \cap U_i \cap A$.

It follows from the procedure above that the total weight of uncolored vertices is never increased, and the weight of an uncolored vertex $v \in U_i$ has weight $w(v) = 2^{m_i(v)}$, where $m_i(v) = |L(v) \cap \{1, 2, \ldots, i-1\}|$. As the total weight of $V(K_{n,n})$ is $2n$, we conclude that each uncolored vertex v has weight at most $2n$. Since $m_{q+1}(v) = k$, after the qth step, if v is still uncolored, then $w(v) = 2^k > 2n$, which is a contradiction. Thus after the qth step, all vertices are colored and we obtain an L-coloring of G. This proves that $ch(G) \leq \log_2 n + 2$. ∎

As $\chi(K_{n,n}) = 2$, the difference $ch(K_{n,n}) - \chi(K_{n,n})$ can also be arbitraruly large.

The next lemma shows that $AT(G)$ is a monotone graph invariant, i.e., if H is a subgraph of G, then $AT(H) \leq AT(G)$.

Lemma 2.1.6 *Assume G is f-AT and $e = uv \in E(G)$. Then $G-e$ is either $(f-1_u)$-AT or $(f-1_v)$-AT. In particular, $G-e$ is f-AT. As a consequence, for any subgraph H of G, $AT(H) \leq AT(G)$.*

Proof. Let $H = G - e$. By definition,

$$P_G = (x_u - x_v)P_H = x_u P_H - x_v P_H.$$

Thus for any $g \in \mathbb{N}^G$,

$$c_{G,g} = c_{H,g-1_u} - c_{H,g-1_v}.$$

As G is f-AT, there exists $g \in \mathbb{N}^G$ such that $c_{G,g} \neq 0$ and $g(w) < f(w)$ for every vertex w of G. Thus $c_{H,g-1_u} - c_{H,g-1_v} \neq 0$, and hence at least one of $c_{H,g-1_u}$ and $c_{H,g-1_v}$ is non-zero. Hence $H = G - e$ is either $(f-1_u)$-AT or $(f-1_v)$-AT. ∎

2.2 Alon-Tarsi Orientation

It is in general a difficult task to determine or estimate the AT number of a graph. Nevertheless, for some families of graphs, there are tools that can be used to derive very tight upper bound for their AT numbers. One of the tools is Alon-Tarsi Theorem, which transforms the calculation of the coefficient of a monomial of P_G into counting the number of even and odd Eulerian sub-digraphs of an orientation of G.

Definition 2.2.1 *A digraph D is called Eulerian if for every vertex v, $d^+_D(v) = d^-_D(v)$.*

Definition 2.2.2 *Given a digraph D, let*

$$\mathcal{E}(D) = \{H \subseteq E(D) : H \text{ induces a Eulerian sub-digraph of } D\ \}.$$

For a subset $\mathcal{F}$ of $\mathcal{E}(D)$, let

$$\begin{aligned}
\mathcal{E}_e(\mathcal{F}) &= \{H \in \mathcal{F} : |H| \equiv 0 \pmod 2\}, \\
\mathcal{E}_o(\mathcal{F}) &= \{H \in \mathcal{F} : |H| \equiv 1 \pmod 2\}, \\
\mathrm{diff}(\mathcal{F}) &= |\mathcal{E}_e(\mathcal{F})| - |\mathcal{E}_o(\mathcal{F})|.
\end{aligned}$$

Let

$$\mathcal{E}_e(D) = \mathcal{E}_e(\mathcal{E}(D)), \mathcal{E}_o(D) = \mathcal{E}_o(\mathcal{E}(D)), \mathrm{diff}(D) = \mathrm{diff}(\mathcal{E}(D)).$$

An element $H \in \mathcal{E}(D)$ is called a Eulerain sub-digraph of D. On the other hand, H is treated as a subset of $E(D)$, so we can take operations on set, such as the union, the intersection, etc., on elements in $\mathcal{E}$.

Observation 1 *If $\mathcal{F}$ is partitioned into $\mathcal{F}_1 \cup \mathcal{F}_2$, then* $\mathrm{diff}(\mathcal{F}) = \mathrm{diff}(\mathcal{F}_1) + \mathrm{diff}(\mathcal{F}_2)$.

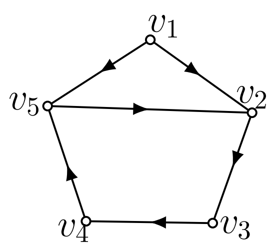

Figure 2.2: A digraph D with $\mathrm{diff}(D) = 2$

Note that the empty set $\emptyset \in \mathcal{E}_e(D)$ for any digraph D. So $\mathcal{E}_e(D) \neq \emptyset$ for any digraph D. The following easy observation will be frequently used in later proofs.

Observation 2 *The arc set of a Eulerian digraph can be decomposed into directed cycles.*

Indeed, if D is a Eulerian digraph, and $e = (u, v) \in E(D)$, then $d^-_D(v) \geq 1$ implies that $d^+_D(v) \geq 1$. Repeating this argument, we obtain a closed directed walk containing e. So e is contained in a directed cycle C. Now $D - E(C)$ is also a Eulerian digraph. Repeating this process, we decompose $E(D)$ into disjoint union of directed cycles.

It follows from Observation 2 that $\mathcal{E}_o(D) = \emptyset$ if and only if D contains no odd directed cycles. In particular, if D is an acyclic digraph (i.e., it does not contain a directed cycle), then $\mathcal{E}_o(D) = \emptyset$ and $\mathcal{E}_e(D) = \{\emptyset\}$.

Assume that D is an orientation of a graph G. For a subset A of arcs of D, denote by A^R the set of arcs obtained by reversing the orientation of the arcs in A. For convenience, a digraph is viewed as a set of arcs. Thus $(D - A) \cup A^R$ denotes the digraph obtained from D by reversing the direction of the arcs in A. Note that every orientation D' of G is of the form $(D - A) \cup A^R$, where A is the set of arcs of D that are reversed in D'.

Lemma 2.2.3 *Assume D and D' are orientations of a graph G. Then $d^+_{D'}(v) = d^+_D(v)$ for all $v \in V(G)$ if and only if $D' = (D - A) \cup A^R$ for some $A \in \mathcal{E}(D)$.*

Proof. Assume $D' = (D-A) \cup A^R$. For any vertex v, $d^+_{D'}(v) = d^+_D(v) + d^+_A(v) - d^-_A(v)$. Therefore $d^+_{D'}(v) = d^+_D(v)$ for all $v \in V(G)$ if and only if $A \in \mathcal{E}(D)$. ∎

Lemma 2.2.4 *Assume G is a graph and D is an orientation of G. Let $g(v) = d^+_D(v)$ for each vertex v. Then*

$$|c_{G,g}| = |\mathrm{diff}(D)|.$$

Proof. We may assume that $P_G(x_v : v \in V(G)) = \prod_{(u,v)\in E(D)}(x_u - x_v)$, and we shall prove that $c_{G,g} = \mathrm{diff}(D)$.

In calculating the expansion of P_G, we first do $2^{|E(G)|}$ multiplications, where each multiplication is done by choosing, for each arc $(u, v) \in E(D)$, either x_u or $-x_v$ participates in the multiplication. We use an orientation D' of G to indicate for each arc (u, v) of D, whether x_u or $-x_v$ is chosen.

For an arc $e = (u, v)$ of D, we define the weight $w(e)$ to be $w(e) = x_u$ if $(u, v) \in E(D')$ and $w(e) = -x_v$ if $(v, u) \in E(D')$, and let $w(D') = \prod_{e\in A(D)} w(e)$. Then

$$P_G(x_v : v \in V(G)) = \sum_{D'} w(D'), \tag{2.1}$$

where the summation ranges over all the $2^{|E(G)|}$ orientations D' of G.

Assume that D' is an orientation of G. Let A be the set of arcs of D that are reversed in D', i.e., $D' = (D - A) \cup A^R$. Then

$$w(D') = (-1)^{|A|} \prod_{v\in V(G)} x_v^{d^+_{D'}(v)}.$$

So D' makes a contribution of 1 or -1 (depending on the parity of $m(D')$) to the coefficient $\mathbf{x}^g$ if and only if $d^+_{D'}(v) = g(v) = d^+_D(v)$ for each vertex v. By Lemma 2.2.3, $A \in \mathcal{E}(D)$. Hence $c_{G,g} = \sum_{A\in\mathcal{E}(D)}(-1)^{|A|} = |\mathcal{E}_e(D)| - |\mathcal{E}_o(D)| = \mathrm{diff}(D)$. ∎

Definition 2.2.5 *Assume D is an orientation of G. If* $\mathrm{diff}(D) \neq 0$*, then we say D is an Alon-Tarsi orientation (or an AT-orientation, for short) of G.*

For a digraph D, let $\Delta_D^+ = \max\{d_D^+(v) : v \in V(D)\}$ be the maximum out-degree of D. The following is a corollary of Lemma 2.2.4.

Corollary 2.2.6 *A graph G is f-AT if and only if G has an AT-orientation D with $d_D^+(v) < f(v)$ for all $v \in V(G)$, and*

$$AT(G) = \min\{k : \ G \text{ has an AT-orientation } D \text{ with } \Delta_D^+ = k-1\}.$$

Lemma 2.2.7 *For any two orientations D_1, D_2 of a graph G with the same out-degree sequence, $|\mathcal{E}(D_1)| = |\mathcal{E}(D_2)|$ and $|\mathrm{diff}(D_1)| = |\mathrm{diff}(\mathrm{D}_2)|$.*

Proof. Assume D_1, D_2 are orientations of G with the same out-degree sequence. Then $D_2 = (D_1 - A) \cup A^R$ for some $A \in \mathcal{E}(D_1)$. For any $H \in \mathcal{E}(D_1)$, let $\phi(H) = (H - A) \cup (H^R \cap A)$. Then ϕ is a bijection from $\mathcal{E}(D_1)$ to $\mathcal{E}(D_2)$. Moreover, $|H| \equiv |\phi(H)| + |A| \pmod 2$. Therefore $|\mathcal{E}(D_1)| = |\mathcal{E}(D_2)|$ and $|\mathrm{diff}(D_1)| = |\mathrm{diff}(\mathrm{D}_2)|$. ∎

Definition 2.2.8 *A digraph D is* strongly connected *if for any two vertices u and v, there is a directed path from u to v.*

Definition 2.2.9 *For a digraph D, a* strongly connected component *of D is a maximal sub-digraph of D which is strongly connected.*

Observe that for any digraph D, there is a partial ordering $\leq$ of the strongly connected components of D such that for any two strongly connected components A and B of D, if $A < B$, then there are arcs between A and B and all the arcs between A and B are from A to B.

Lemma 2.2.10 *Assume that D is a digraph and $V(D) = X_1 \dot{\cup} X_2$. For $i = 1, 2$, let $D_i = D[X_i]$ be the sub-digraph of D induced by X_i. If all the arcs between X_1 and X_2 are from X_1 to X_2, then D is Alon-Tarsi if and only if D_1, D_2 are both Alon-Tarsi.*

Proof. Since any arc of a Eulerian digraph is contained in a directed cycle, each Eulerian sub-digraph H of D is the arc disjoint union of H_1 and H_2, where H_i is a Eulerian sub-digraph of D_i. Now H is even if and only if H_1, H_2 have the same parity. Therefore

$$\begin{aligned}
|\mathcal{E}_e(D)| &= |\mathcal{E}_e(D_1)| \times |\mathcal{E}_e(D_2)| + |\mathcal{E}_o(D_1)| \times |\mathcal{E}_o(D_2)|, \\
|\mathcal{E}_o(D)| &= |\mathcal{E}_e(D_1)| \times |\mathcal{E}_o(D_2)| + |\mathcal{E}_o(D_1)| \times |\mathcal{E}_e(D_2)|.
\end{aligned}$$

So $\mathrm{diff}(D) = \mathrm{diff}(D_1) \times \mathrm{diff}(D_2)$ and D is AT if and only if both D_1, D_2 are AT. ∎

Corollary 2.2.11 *Assume D is a digraph and $V(D) = X_1 \cup X_2$, and $D_i = D[X_i]$ for $i = 1, 2$. If for all vertices $v \in X_1 \cap X_2$, $d^+_{D_2}(v) = 0$, then D is Alon-Tarsi if and only if D_1, D_2 are both Alon-Tarsi.*

Proof. Let $X'_2 = X_2 - X_1$. Then $V(D) = X_1 \dot{\cup} X'_2$, and all the edges between X_1 and X'_2 are oriented from X'_2 to X_1. Also D_2 is Alon-Tarsi if and only if $D[X'_2]$ is Alon-Tarsi. ∎

Corollary 2.2.12 *A digraph is Alon-Tarsi if and only if each of its strongly connected components is Alon-Tarsi.*

Lemma 2.2.13 *Assume D is a digraph and $V(D) = X_1 \cup X_2$, and $D_i = D[X_i]$ for $i = 1, 2$. If $|X_1 \cap X_2| = 1$, then D is Alon-Tarsi if and only if D_1, D_2 are both Alon-Tarsi.*

Proof. For any sub-digraph H of D, let $H_i = H \cap D_i$. Then H is Eulerian if and only if both H_1 and H_2 are Eulerian. Hence $\mathrm{diff}(D) = \mathrm{diff}(D_1) \times \mathrm{diff}(D_2)$ and D is AT if and only if both D_1, D_2 are AT. ∎

2.3 Bipartite graphs and acyclic orientations

In general, it is difficult to determine if an orientation D of a graph G is AT. However, there are some cases where this problem is easy. Observe that every digraph D has at least one even Eulerian sub-digraph, namely, the empty sub-digraph, and every Eulerian digraph can be decomposed into directed cycles. So if D has no odd directed cycle, then D has no odd Eulerian sub-digraph and hence D is AT. Thus we have the following result.

Lemma 2.3.1 *If G is a bipartite graph, then every orientation D of G is an AT-orientation. For any graph G, any acyclic orientation D of G is an AT-orientation of G.*

Corollary 2.3.2 *If G is a bipartite graph, then $AT(G) = \max_{H \subset G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil + 1$.*

Proof. Assume that $k = \max_{H \subset G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil$. By Theorem 2.1.4, $AT(G) \geq k + 1$.

We now show that there is an orientation D of G for which $\Delta^+_D \leq k$. We construct a bipartite graph Q with partite sets $E(G)$ and $V(G) \times [k]$, where $[k] = \{1, 2, ..., k\}$, and $(v, j) \in V(G) \times [k]$ is adjacent to $e \in E(G)$ if v is an end vertex of e. For any subset X of $E(G)$, let Y be the set of vertices of G each of which is incident to at least one edge of X. Then $N_Q(X) = Y \times [k]$. Since $|X| \leq k|Y|$, we conclude that $|X| \leq |N_Q(X)|$. By Hall's Theorem, Q has a matching M that saturates every $e \in E$. We orient the edge $e = uv \in E(G)$ as (u, v) if and only if $(e, (u, j)) \in M$ for some j. Then the resulting orientation D has $\Delta^+_D \leq k$. By Corollary 2.2.6, $AT(G) \leq k + 1$. Hence $AT(G) = k + 1$. ∎

Corollary 2.3.3 *Bipartite planar graphs G have $AT(G) \leq 3$ and are therefore 3-choosable.*

Proof. By Euler's formula, for any connected subgraph H of G, $|V(H)| + |F(H)| - |E(H)| = 2$. As each face of H has degree at least 4, we conclude that $4|F(H)| \leq \sum_{f \in F(H)} d(f) = 2|E(H)|$. So $|F(H)| \leq \frac{1}{2}|E(H)|$. This implies $|E(H)|/|V(H)| < 2$. ∎

Definition 2.3.4 *Assume G is a graph and $f \in \mathbb{N}^G$. We say G is* strict f-degenerate *if G has an acyclic orientation D such that $d^+_D(v) < f(v)$ for all $v \in V(G)$. The* strict degeneracy *(also called the coloring number) of G, denoted by $st(G)$ (also denoted by* $\mathrm{col}(G)$*), is defined as*

$$st(G) = \min\{k : G \textit{ is strict } k\textit{-degenerate}\}.$$

Lemma 2.3.5 *For any graph G,*

$$st(G) = 1 + \max\{\delta(H) : H \textit{ is a subgraph of } G\},$$

where $\delta(H)$ stands for the minimum degree of H.

Alternately, $st(G)$ is the minimum integer k such that there is a linear ordering $<$ of the vertices of G so that each vertex v has at most $k - 1$ neighbors u with $u < v$. By orienting each edge uv as (u, v) if $u > v$, we obtain an acyclic orientation D of G with $\delta^+_D = \mathrm{col}(G) - 1$. Hence we have the following corollary.

Corollary 2.3.6 *For any graph G, $AT(G) \leq st(G)$.*

2.4 The Cartesian product of a path and an odd cycle

Another simple sufficient condition for an orientation D to be Alon-Tarsi is that $|\mathcal{E}(D)|$ is odd. For, if $|\mathcal{E}_e(D)| = |\mathcal{E}_o(D)|$, then $|\mathcal{E}(D)| = |\mathcal{E}_e(D)| + |\mathcal{E}_o(D)|$ is even. Hence if $|\mathcal{E}(D)|$ is odd, then $|\mathcal{E}_e(D)| \neq |\mathcal{E}_o(D)|$ and D is Alon-Tarsi. In this section, this idea is used to prove that the Cartesian product of a path and an odd cycle has Alon-Tarsi number 3.

Definition 2.4.1 *Assume that $G = (V, E)$ and $G' = (V', E')$ are graphs. The* Cartesian product *$G \Box G'$ is the graph with vertex set $V \times V'$, in which (x, x') is adjacent to (y, y') if and only if either $x = x'$ and $yy' \in E'$ or $y = y'$ and $xx' \in E$.*

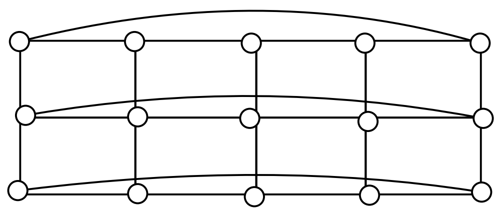

Figure 2.3: The graph $P_3 \Box C_5$

Theorem 2.4.2 *[18] If G is the Cartesian product $C_{2k+1} \Box P_n$ of an odd cycle and a path, then $AT(G) = 3$.*

Proof. Since G has chromatic number 3, we know that $AT(G) \geq 3$. It remains to show that $AT(G) \leq 3$. If $n = 1$, then the result is obvious. We assume that $n \geq 2$.

Assume that the vertices of G are $\{(v_i, w_j) : 1 \leq i \leq 2k+1, 1 \leq j \leq n\}$, where for each j, the set $X_j = \{(v_i, w_j) : 1 \leq i \leq 2k+1\}$ induces an odd cycle of length $2k+1$, and for each i, the set $Y_i = \{(v_i, w_j) : 1 \leq j \leq n\}$ induces a path on n vertices. Let D be the orientation of G in which the edges of G are oriented in such a way that for each j, $G[X_j]$ is a directed cycle, and for each i, $G[Y_i]$ is a directed path from (v_i, w_1) to (v_i, w_n) [See Figure 2.4].

Let D^* be obtained from D by adding an arc e^* from (v_2, w_n) to (v_1, w_1). It is easy to check that D^* has maximum out-degree 2. Now we show that D^* has an odd number of Eulerian sub-digraphs.

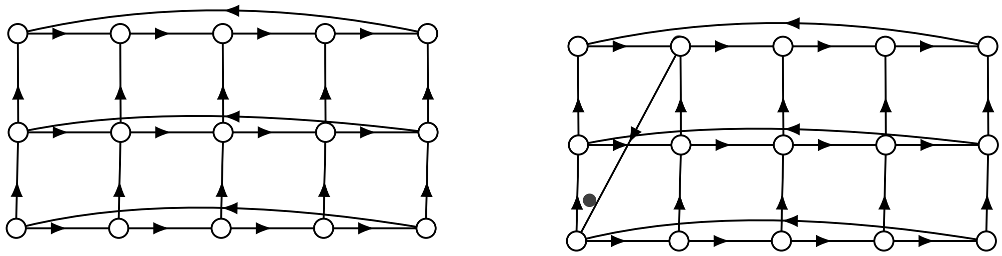

Figure 2.4: The digraphs D and D^* for $n = 3$ and $k = 2$

Note that the only directed cycles in D are those directed cycles induced by X_j for $j = 1, 2, \ldots, n$. All the directed cycles in D^* not contained in D contain the arc e^*.

Let

$$\mathcal{A} = \{Q \in \mathcal{E}(D^*) : \exists j \in \{1, 2, \ldots, n\}, E(D[X_j]) \subseteq E(Q) \text{ or } E(D[X_j]) \cap E(Q) = \emptyset\}.$$

For $Q \in \mathcal{A}$, let $\pi(Q) = Q \Delta E(D[X_j])$, where j is the smallest index for which $E(D[X_j]) \subseteq E(Q)$ or $E(D[X_j]) \cap E(Q) = \emptyset$. Then $Q \neq \pi(Q) \in \mathcal{A}$ and $\pi^2(Q) = Q$. So $|\mathcal{A}|$ is even.

Let $\mathcal{B}$ be the set of all the other Eulerian sub-digraphs of D^*. Since each Eulerian sub-digraph can be decomposed into an arc-disjoint union of directed cycles (and the only directed cycles in D are $D[X_j]$, for $1 \leq j \leq n$), we conclude that each member of $\mathcal{B}$ is a directed cycle containing e^* and contains at least one arc and at most $2k$ arcs from each directed cycle $D[X_j]$. To construct such a directed cycle C, we start from (v_1, w_1), traverse at least one arc but at most $2k$ arcs from $D[X_1]$, then go to X_2 and traverse at least one arc but at most $2k$ arcs from $D[X_2]$, and then continue in the same fashion to $X_3, X_4, \ldots, X_n$. Such a directed cycle is uniquely determined by the positions in which it goes from X_i to X_{i+1} for $i = 1, 2, \ldots, n-1$. Since the last arc of the directed cycle is from (v_2, w_n) to (v_1, w_1), for the directed cycle to contain at least one and at most $2k$ arcs of $D[X_n]$, it is required that the arc from X_{n-1} to X_n is not the arc joining (v_2, w_{n-1}) to (v_2, w_n).

Let S_n, S'_n and S''_n be sets of integer sequences defined as follows:

$$\begin{aligned}
S_n &= \{(i_1, i_2, \ldots, i_{n-1}) : 1 \leq i_j \leq 2k+1, i_1 \neq 1, i_{j+1} \neq i_j\}, \\
S'_n &= \{(i_1, i_2, \ldots, i_{n-1}) \in S_n : i_{n-1} \neq 2\}, \\
S''_n &= \{(i_1, i_2, \ldots, i_{n-1}) \in S_n : i_{n-1} = 2\}.
\end{aligned}$$

By the argument above, $|\mathcal{B}| = |S'_n|$.

Each sequence in S''_{n-1} can be extended to a sequence in S'_n in $2k$ different ways (i_{n-1} can be any index distinct from i_{n-2}), and each sequence in S'_{n-1} can be extended to a sequence in S'_n in $2k-1$ different ways (i_{n-1} can be any index distinct from both i_{n-2} and 2). So $|S'_n| = |S''_{n-1}|2k + |S'_{n-1}|(2k-1)$. Thus $|S'_n| \equiv |S'_{n-1}| \pmod 2$. As $S'_2 = 2k-1$ is odd, we conclude that $|\mathcal{B}| = |S'_n|$ is odd. Thus D^* is an AT-orientation of $G^* = G+e$, where e is an edge connecting (v_1, w_2) to (v_2, w_n). Hence $AT(G^*) \leq 3$, and by Lemma 2.1.6, $AT(G) \leq 3$. ∎

Corollary 2.4.3 *If $G = C_{2k+1} \Box P_n$ for some positive integers k, n, then $\chi(G) = ch(G) = AT(G) = 3$.*

If $G = C_{2k} \Box P_n$ and $n \geq 2$, then G is a bipartite graph with $1 < |E(G)|/|V(G)| < 2$. By Corollary 2.3.2, $AT(G) = 3$. So the Cartesian product of any cycle with a path with at least two vertices has Alon-Tarsi number 3.

It is natural to ask as to what is the Alon-Tarsi number of the Cartesian product $C_n \Box C_m$ of two cycles. If both n and m are even, then $C_n \Box C_m$ is bipartite, and it follows from Corollary 2.3.2 that $C_n \Box C_m$ has Alon-Tarsi number 3. When at least one of n, m is odd, this question will be answered in Section 4.5.

2.5 A Hamilton cycle plus a family of triangles

Similar to the idea in the previous section, if $|\mathcal{E}_e(D)| = |\mathcal{E}_o(D)|$ is even, then $|\mathcal{E}(D)| \equiv 0 \pmod 4$. Thus if $|\mathcal{E}(D)| \equiv 2 \pmod 4$ and $|\mathcal{E}_e(D)|$ is even, then D is an Alon-Tarsi orientation. In this section, this technique is used to prove a family of 4-regular graphs has Alon-Tarsi number 3.

Assume that D is an Eulerian digraph. For any Eulerian subdigraph H of D, its complement $D \backslash H$ is also an Eulerian digraph. If D has an odd number of arcs, then H and $D \backslash H$ have opposite parity. Hence $|\mathcal{E}_e(D)| = |\mathcal{E}_o(D)|$. So D is not an AT-orientation. On the other hand, if D has an even number of arcs, then H and $D \backslash H$ have the same parity. Thus, $|\mathcal{E}_e(D)|$ and $|\mathcal{E}_o(D)|$ are even. Therefore, if $|\mathcal{E}(D)| \equiv 2 \pmod 4$, then $|\mathcal{E}_e(D)| \neq |\mathcal{E}_o(D)|$. Hence D is an Alon-Tarsi orientation.

We shall frequently use the following lemma.

Lemma 2.5.1 *If H and H' are Eulerian digraphs on the same set of vertices, then $(H - H') \cup (H' - H)^R$ is a Eulerian digraph.*

Proof. In the formula $(H - H') \cup (H' - H)^R$, H and H' are treated as sets of arcs.

Let $H'' = (H - H') \cup (H' - H)^R$. Then

$$\begin{aligned} E^+_{H''}(v) &= E^+_H(v) - E^+_{H\cap H'}(v) + E^-_{H'}(v) - E^-_{H\cap H'}(v), \\ E^-_{H''}(v) &= E^-_H(v) - E^-_{H\cap H'}(v) + E^+_{H'}(v) - E^+_{H\cap H'}(v). \end{aligned}$$

As $d^+_H(v) = |E^+_H(v)| = |E^-_H(v)| = d^-_H(v)$ and $d^+_{H'}(v) = |E^+_{H'}(v)| = |E^-_{H'}(v)| = d^-_{H'}(v)$, we conclude that $d^+_{H''}(v) = d^-_{H''}(v)$. So H'' is a Eulerain digraph. ∎

Assume that G is a 4-regular graph on $3n$ vertices whose edge set can be decomposed into a Hamiltonian cycle and n pairwise vertex-disjoint triangles. Erdős asked whether G is 3-colorable. This question, often referred to as "Erdős' cycle plus triangle problem", was answered in the affirmative by Fleischner and Stiebitz in 1992 [15]. Indeed, Fleischner and Stiebitz proved the following stronger result.

Theorem 2.5.2 *Assume that G is a 4-regular graph on $3n$ vertices whose edge set can be decomposed into a Hamiltonian cycle and n pairwise vertex disjoint triangles. Then $AT(G) = 3$.*

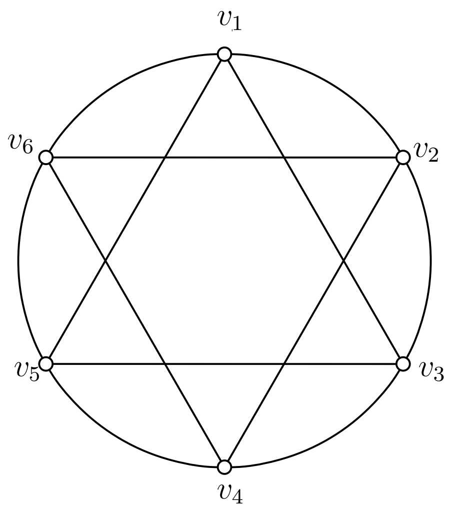

Figure 2.5: An example of a graph G of Theorem **??**

To prove Theorem **??**, we need to find an AT-orientaiton D of G in which each vertex has out-degree at most 2. Since $\sum_{x\in V(D)} d^+_D(x) = |A(D)| = 2|V(D)|$, we conclude that $d^+_D(x) = d^-_D(x) = 2$ for each vertex x. So D is an Eulerian digraph with an even number of arcs. Hence Theorem **??** follows from the next lemma.

Lemma 2.5.3 *Assume that D is an Eulerian digraph whose edge set can be decomposed into a directed Hamiltonian cycle C and $n \geq 0$ pairwise vertex-disjoint directed triangles. Then $|\mathcal{E}(D)| \equiv 2 \pmod 4$.*

Proof. As the n triangles are vertex disjoint, $|V(D)| \geq 3n$. We observe that the number of vertices of D can be greater than n. Thus some vertices of D may not be contained in any of the n triangles, and hence has in-degree and out-degree both equal to 1.

We prove the lemma by induction on n. If $n = 0$, then D is a directed cycle and $|\mathcal{E}(D)| = 2$ (the only Eulerian subdigraphs of D are the empty digraph and D itself).

Assume that $n \geq 1$ and the Lemma holds if there are at most $n-1$ triangles.

Let T be one of the triangles as indicated in Figure 2.6, where $V(T) = \{x_1, x_2, x_3\}$ and the arcs of T are $a_1 = (x_3, x_2), a_2 = (x_1, x_3), a_3 = (x_2, x_1)$.

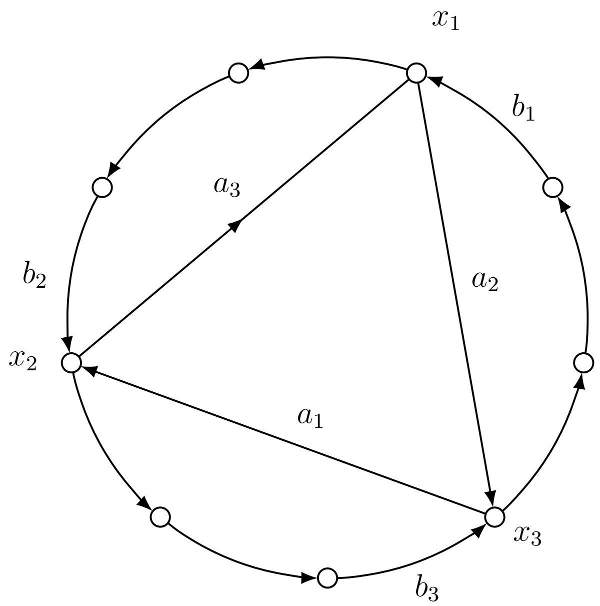

Figure 2.6: Digraph D

Now an Eulerian subdigraph of D may contain one, two, all or none of the arcs a_1, a_2, a_3. Recall that for arcs $z_1, z_2, \ldots, z_q, y_1, y_2, \ldots, y_p$, $\mathcal{E}(D, z_1, z_2, \ldots, z_q, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_p)$ is the family of Eulerian subdigraphs H of D which contain $z_1, z_2, \ldots, z_q$ but which do not contain any of $y_1, y_2, \ldots, y_p$.

Hence

$$\mathcal{E}(D) \;=\; \mathcal{E}^*(D)\dot{\cup}\mathcal{E}(D, a_1, a_2, a_3)\dot{\cup}\mathcal{E}(D, \overline{a}_1, \overline{a}_2, \overline{a}_3),$$

where

$$\begin{aligned}\mathcal{E}^*(D) &= \mathcal{E}(D,\overline{a}_1,a_2,a_3)\dot\cup\mathcal{E}(D,\overline{a}_2,a_1,a_3)\dot\cup\mathcal{E}(D,\overline{a}_3,a_1,a_2)\\ &\quad \dot\cup\mathcal{E}(D,a_1,\overline{a}_2,\overline{a}_3)\dot\cup\mathcal{E}(D,a_2,\overline{a}_1,\overline{a}_3)\dot\cup\mathcal{E}(D,a_3,\overline{a}_1,\overline{a}_2).\end{aligned}$$

Let $D' = D - A(T)$, where $A(T)$ stands for the arc set of T. Then D' is an Eulerian digraph whose arc set is decomposed into a Hamiltonian cycle and $n-1$ pairwise vertex disjoint triangles. By induction hypothesis, $|\mathcal{E}(D')| \equiv 2 \pmod 4$. It is obvious that

$$|\mathcal{E}(D')| = |\mathcal{E}(D,a_1,a_2,a_3)| = |\mathcal{E}(D,\overline{a}_1,\overline{a}_2,\overline{a}_3)|.$$

So $|\mathcal{E}(D,a_1,a_2,a_3)| + |\mathcal{E}(D,\overline{a}_1,\overline{a}_2,\overline{a}_3)| \equiv 0 \pmod 4$. As

$$|\mathcal{E}(D)| = |\mathcal{E}^*(D)| + |\mathcal{E}(D,a_1,a_2,a_3)| + |\mathcal{E}(D,\overline{a}_1,\overline{a}_2,\overline{a}_3)|,$$

we have $|\mathcal{E}(D)| \equiv |\mathcal{E}^*(D)| \pmod 4$. So it remains to show that $|\mathcal{E}^*(D)| \equiv 2 \pmod 4$.

For $i = 1,2,3$, let x_i^- be the predecessor of x_i on C and let $b_i = (x_i^-, x_i)$. By Lemma 2.2.3, we may assume that the cycle C is directed as in Figure 2.6. For $i = 1,2,3$, let C_i be the unique directed cycle in $C \cup a_i$ containing a_i.

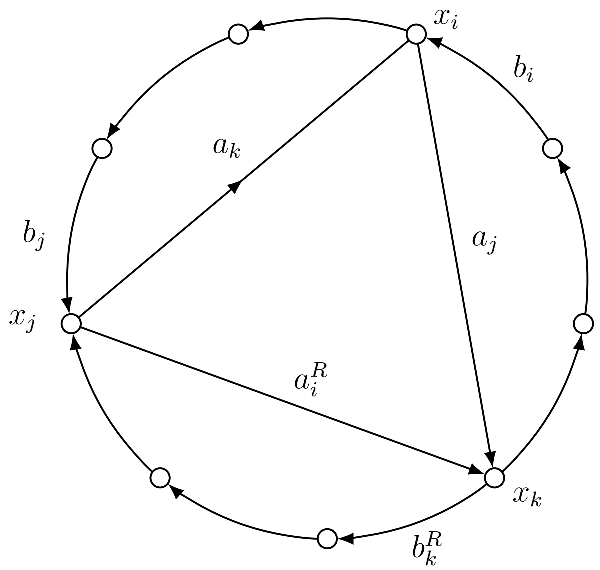

Figure 2.7: Digraph D_i

For $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$, let $D_i = (D - C_i) \cup C_i^R$ as in Figure 2.7 and let D_i' be obtained from D_i by splitting away arcs b_j, a_k, a_j and b_k^R as in Figure 2.8. Let x_i', x_j', x_k' be the new vertices that result from the splitting. Then D_i' is an

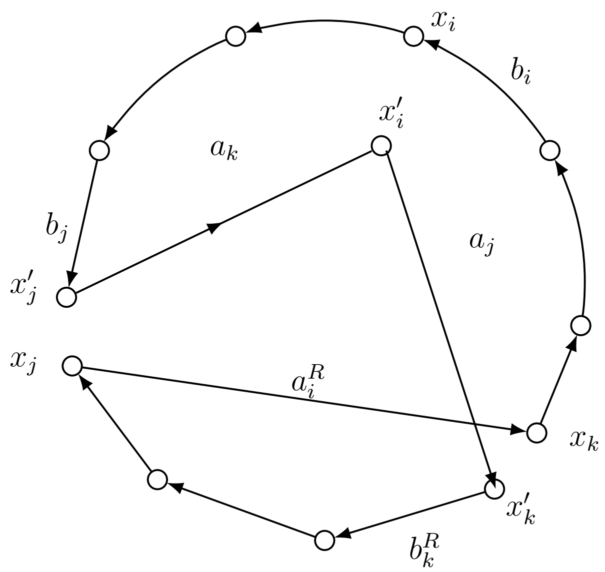

Figure 2.8: Digraph D'_i

Eulerian digraph which has a decompsotion into a directed Hamiltonian cycle and $(n-1)$ pairwise vertex disjoint triangles.

Since a_j and a_k share a degree 2 vertex, each Eulerian subdigraph H of D'_i either contains both a_j and a_k, or contains neither of a_j and a_k. Hence

$$|\mathcal{E}(D'_i)| = |\mathcal{E}(D'_i, a_i^R, a_j, a_k)| + |\mathcal{E}(D'_i, \overline{a}_i^R, a_j, a_k)| + |\mathcal{E}(D'_i, a_i^R, \overline{a}_j, \overline{a}_k)| + |\mathcal{E}(D'_i, \overline{a}_i^R, \overline{a}_j, \overline{a}_k)|.$$

Claim 2.5.4 *For* $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$,

$$\begin{aligned}
|\mathcal{E}(D'_i, a_i^R, a_j, a_k)| &= |\mathcal{E}(D, \overline{a}_i, a_j, a_k)|, \\
|\mathcal{E}(D'_i, \overline{a}_i^R, \overline{a}_j, \overline{a}_k)| &= |\mathcal{E}(D, a_i, \overline{a}_j, \overline{a}_k)|, \\
|\mathcal{E}(D'_i, a_i^R, \overline{a}_j, \overline{a}_k)| &= |\mathcal{E}(D', \overline{b}_j, b_k)| = |\mathcal{E}(D', b_i, \overline{b}_j, b_k)| + |\mathcal{E}(D', \overline{b}_i, \overline{b}_j, b_k)|, \\
|\mathcal{E}(D'_i, \overline{a}_i^R, a_j, a_k)| &= |\mathcal{E}(D', b_j, \overline{b}_k)| = |\mathcal{E}(D', b_i, b_j, \overline{b}_k)| + |\mathcal{E}(D', \overline{b}_i, b_j, \overline{b}_k)|.
\end{aligned}$$

Proof. For $H \in \mathcal{E}(D)$, let $\phi(H) = (H - C_i) \cup (C_i - H)^R$. By Lemma 2.5.1, $\phi(H) \in \mathcal{E}(D'_i)$. It is easy to see that ϕ is a one-to-one correspondence between $\mathcal{E}(D, \overline{a}_i, a_j, a_k)$ and $\mathcal{E}(D'_i, a_i^R, a_j, a_k)$, and a one-to-one correspondence between $\mathcal{E}(D, a_i, \overline{a}_j, \overline{a}_k)$ and $\mathcal{E}(D'_i, \overline{a}_i^R, \overline{a}_j, \overline{a}_k)$.

For $H \in \mathcal{E}(D'_i)$, $\psi(H) = (H - C_i^R) \cup (C_i^R - H)^R$ is a one-to-one correspondence between $\mathcal{E}(D'_i, a_i^R, \overline{a}_j, \overline{a}_k)$ and $\mathcal{E}(D', \overline{b}_j, b_k)$, and $\psi'(H) = \overline{\psi(H)} - \{a_i, a_j, a_k\}$ is a one-to-one correspondence between $\mathcal{E}(D'_i, \overline{a}_i^R, a_j, a_k)$ and $\mathcal{E}(D', b_j, \overline{b}_k)$. ∎

It follows from Claim 2.5.4 that

$$|\mathcal{E}(D'_1)| + |\mathcal{E}(D'_2)| + |\mathcal{E}(D'_3)| \quad = \quad |\mathcal{E}^*(D)| + m,$$

where

$$m \quad = \quad \sum_{(i,j,k)} \left(|\mathcal{E}(D', b_i, \overline{b}_j, b_k)| + |\mathcal{E}(D', \overline{b}_i, \overline{b}_j, b_k)| + |\mathcal{E}(D', b_i, b_j, \overline{b}_k)| + |\mathcal{E}(D', \overline{b}_i, b_j, \overline{b}_k)|\right),$$

where the summation is over $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

By induction hypothesis, $|\mathcal{E}(D'_i)| \equiv 2 \pmod 4$ and hence

$$|\mathcal{E}(D'_1)| + |\mathcal{E}(D'_2)| + |\mathcal{E}(D'_3)| \equiv 2 \pmod 4.$$

To prove that $|\mathcal{E}^*(D) \equiv 2 \pmod 4$, it remains to show that $m \equiv 0 \pmod 4$.

By using the one-to-one correspondence $\phi : \mathcal{E}(D') \to \mathcal{E}(D')$ defined as $\phi(H) = D \backslash H$, we conclude that $|\mathcal{E}(D', b_i, \overline{b}_j, \overline{b}_k)| = |\mathcal{E}(D', \overline{b}_i, b_j, b_k)|$. It follows that

$$\begin{aligned} & |\mathcal{E}(D', b_i, \overline{b}_j, b_k)| + |\mathcal{E}(D', \overline{b}_i, \overline{b}_j, b_k)| + |\mathcal{E}(D', b_i, b_j, \overline{b}_k)| + |\mathcal{E}(D', \overline{b}_i, b_j, \overline{b}_k)| \\ = \quad & 2(|\mathcal{E}(D', b_i, \overline{b}_j, b_k)| + |\mathcal{E}(D', b_i, b_j, \overline{b}_k)|). \end{aligned}$$

Therefore

$$m = 4(|\mathcal{E}(D', \overline{b}_1, b_2, b_3)| + |\mathcal{E}(D', b_1, \overline{b}_2, b_3)| + |\mathcal{E}(D', b_1, b_2, \overline{b}_3)|) \equiv 0 \pmod 4.$$

This completes the proof of Lemma 2.5.3, and hence completes the proof of Theorem **??**. ∎

Theorem **??** was extended in [14], where it was proved that if $k \geq 4$, and G is a graph whose edge set can be decomposed into a Hamilton cycle and a family of vertex disjoint complete graphs of order at most k, then G is k-colourable. A different proof of Theorem 2.4.2 is given in [27].

Chapter 3

Graph polynomial for edge-weighted graphs

3.1 Arc-weighted orientation

Definition 3.1.1 *An edge-weighted graph, denoted by (G, w), is a graph associated with a weighting function $w : E(G) \to \{1, 2, \ldots\}$ that assigns to each edge of G a positive integer as its weight. An* arc-weighted orientation *of a graph G is a pair (D, w) such that D is an orientation of G, and $w : E(D) \to \{1, 2, \ldots\}$ is a mapping that assigns to each arc of D a positive integer as its weight.*

Equivalently, an arc-weighted orientation of G is an orientation of an edge-weighted graph (G, w). The weight of an arc (u, v) is the weight of the edge uv, and denoted by $w(uv)$. The pair (D, w) is also called an *arc-weighted digraph.* Let

$$P_{(G,w)}(x_v : v \in V(G)) = \prod_{(u,v)\in E(D)} (x_u^{w(uv)} - x_v^{w(uv)}).$$

Note that if $w(uv) = 1$ for all $uv \in E(G)$, then $P_{(G,w)} = P_G$. The definition of $P_{(G,w)}$ depends on the orientation D of G. To indicate which orientation D is used in the definition of $P_{(G,w)}$, we may write $P_{(G,w)}$ as $P_{(D,w)}$. Similarly, for any two orientations D and D' of G, we have $P_{(D,w)} = (-1)^r P_{(D',w)}$, where r is the number of edges that are oriented differently by D and D'.

Since for any positive integer $k \geq 2$,

$$(x^k - y^k) = (x - y)(x^{k-1} + x^{k-2}y + \ldots + y^{k-2}x + y^{k-1}),$$

we conclude that for any weight function w,

$$P_{(G,w)}(x_v : v \in V(G)) = P_G(x_v : v \in V(G)) \cdot q$$

for some polynomial $q \in \mathbb{R}[x_v : v \in V(G)]$ of degree $\sum_{e\in E(G)}(w(e)-1)$. For $g \in \mathbb{N}^G$, let $c_{(G,w),g}$ be the coefficient of the monomial $\mathbf{x}^g$ in the expansion of $P_{(G,w)}$, and let $c_{q,g}$ be the coefficient of the monomial $\mathbf{x}^g$ in the expansion of q. Then

$$c_{(G,w),g} = \sum_{g' \leq g, \sum_{v\in V(G)} g(v) = |E|} c_{G,g'} c_{q,(g-g')}.$$

Therefore if $c_{(G,w),g} \neq 0$, then $c_{G,g'} \neq 0$ for some $g' \in \mathbb{N}^G$ with $g'(v) \leq g(v)$ for all $v \in V(G)$.

Definition 3.1.2 *Assume (D,w) is an arc-weighted orientation of G. For $v \in V(G)$, the out-degree of v in (D,w) is*

$$d^+_{(D,w)}(v) = \sum_{(v,u)\in E(D)} w(vu),$$

and the in-degree of v in (D,w) is

$$d^-_{(D,w)}(v) = \sum_{(u,v)\in E(D)} w(uv).$$

Definition 3.1.3 *We say (D,w) is called Eulerian if for every vertex v, $d^+_{(D,w)}(v) = d^-_{(D,w)}(v)$.*

For a subset H of $E(D)$, we denote by (H,w) the arc-weighted sub-digraph of (D,w) induced by H.

Definition 3.1.4 *Given an arc-weighted digraph (D,w), let*

$$\mathcal{E}(D,w) = \{H \subseteq E(D) : (H,w) \text{ is Eulerian }\}.$$

Let

$$\begin{aligned}
\mathcal{E}_e(D,w) &= \{H \in \mathcal{E}(D,w) : |H| \equiv 0 \pmod 2\},\\
\mathcal{E}_o(D,w) &= \{H \in \mathcal{E}(D,w) : |H| \equiv 1 \pmod 2\},\\
\mathrm{diff}(D,w) &= |\mathcal{E}_e(D,w)| - |\mathcal{E}_o(D,w)|.
\end{aligned}$$

The same argument as the proof of Lemma 2.2.4 proves the following lemma.

Lemma 3.1.5 *Assume (D, w) is an arc-weighted orientation of G. Let $g(v) = d^+_{(D,w)}(v)$ for each vertex v. Then*

$$|c_{(G,w),g}| = |\mathrm{diff}(D, w)|.$$

Definition 3.1.6 *An arc-weighted orientation (D, w) of G is called an Alon-Tarsi orientation (AT-orientation for short) if* $\mathrm{diff}(D, w) \neq 0$.

Note that an AT-orientation D of a graph G is an arc-weighted AT-orientation (D, w) of G, with $w(e) = 1$ for each edge e of G. Thus we have the following lemma.

Lemma 3.1.7 *Assume G is a graph and $f \in \mathbb{N}^G$. Then the following are equivalent.*

1. *G is f-AT.*
2. *G has an AT-orientation D such that $d^+_D(v) < f(v)$ for each vertex v.*
3. *G has an arc-weighted AT-orientation (D, w) such that $d^+_{(D,w)}(v) < f(v)$ for each vertex v.*

3.2 Acyclic arc-weighted orientation

We have observed that an acyclic orientation D is AT. Now we extend the concept of acyclic orientation to acyclic arc-weighted orientation.

Definition 3.2.1 *Assume (D, w) is an arc-weighted digraph. An arc $(u, v) \in E(D)$ is called a* dominating arc *if $w(uv) > d^+_{(D,w)}(v)$. We say (D, w) is an* acyclic arc-weighted digraph *if for any sub-digraph D' of D with $E(D') \neq \emptyset$, (D', w) has a dominating arc.*

If $w(e) = 1$ for all arcs e of D, then (D, w) is acyclic if and only if each non-empty sub-digraph D' of D contains a sink, which is equivalent to D be acyclic.

If (D, w) is an arc-weighted acyclic digraph, then (D, w) contains no non-empty Eulerian arc-weighted sub-digraph, because if $e = (u, v)$ is a dominating arc in (D', w), then $d^+_{(D',w)}(v) < w(u,v) \leq d^-_{(D',w)}(v)$. So (D',w) is not Eulerian. As the empty sub-digraph is always an Eulerian arc-weighted sub-digraph, we conclude that an arc-weighted acyclic orientation (D,w) of a graph G is an arc-weighted AT-orientation of G.

The following lemma follows from the definition.

Lemma 3.2.2 *An arc-weighted digraph (D,w) is acyclic if and only if its arcs can be ordered as $e_1, e_2, \ldots, e_m$ such that e_i is a dominating arc in $(D - \{e_1, \ldots, e_{i-1}\}, w)$.*

Definition 3.2.3 *Assume G is a graph and $f \in \mathbb{N}^G$. We G is* arc-weighted strict f-degenerate *("AW strict f-degenerate" for short) if G has an arc-weighted acyclic orientation (D,w) such that $d^+_{(D,w)}(v) < f(v)$ for all $v \in V(G)$. The* arc-weighted strict degeneracy *of G, denoted by $\mathrm{sd}_{\mathrm{AW}}(G)$, is the minimum integer k such that G is arc-weighted strict k-degenerate.*

As an acyclic orientation D of G is an arc-weighted acyclic orientation (D,w) with $w(e) = 1$ for all edges e, we know that if G if strict f-degenerate, then G is AW strict f-degenerate, and

$$\mathrm{sd}_{\mathrm{AW}}(G) \leq \mathrm{sd}(G).$$

The converse is not true. For any r-regular graph G, we have $\mathrm{sd}(G) = r+1$. Let P be the Petersen graph. Then $st(P) = 4$. The following is an arc-weighted orientation (D,w) of P with $d^+_{(D,w)}(v) \leq 2$ for all $v \in V(G)$, and hence $\mathrm{sd}_{\mathrm{AW}}(P) \leq 3 < st(P)$. It is easy to check the equality $\mathrm{sd}_{\mathrm{AW}}(P) = 3$ holds.

Corollary 3.2.4 *If G is AW strict f-degenerate, then G is f-AT, and hence f-choosable. Hence for any graph G,*

$$ch(G) \leq AT(G) \leq \mathrm{sd}_{\mathrm{AW}}(G).$$

3.3 Degree AT graphs

It is obvious that for any graph G, $\mathrm{col}(G) \leq \Delta(G) + 1$. Moreover if G is connected, and G is not a regular graph, we have $\mathrm{sd}(G) \leq \Delta(G)$. However if G is a regular graph, then $\mathrm{sd}(G) = \Delta(G) + 1$. Corollary 3.3.7 below shows that $\mathrm{sd}_{\mathrm{AW}}(G) \leq \Delta(G)$ even if G is a regular graph, provided that G is neither a complete graph nor a cycle.

A graph G is called *degree choosable* if for any list assignment L for which $|L(v)| = d_G(v)$ for every vertex v, then G is L-colourable. We say that G is *degree AT* if G is

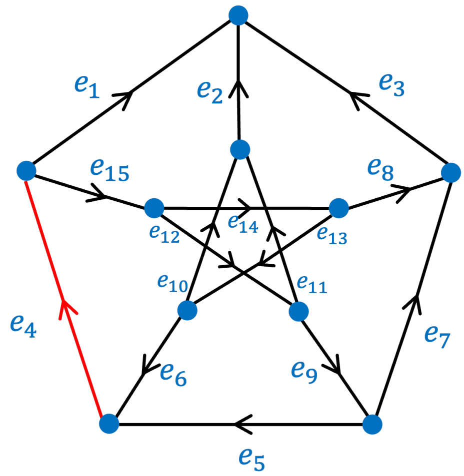

Figure 3.1: Arc-weighted acyclic orientation of the Petersen graph, where $w(e) = 1$ for a black arc and $w(e) = 2$ for a red arc. For each i, e_i is a dominating arc in $D - \{e_1, \dots, e_{i-1}\}$.

f-AT for $f(v) = d_G(v)$ for every vertex v of G. We say that G is *AW strict degree degenerate* if G is AW strict f-degenerate for $f(v) = d_G(v)$ for every vertex v of G.

It follows from the discussion above that if G is AW strict degree degenerate, then G is degree AT; if G is degree AT, then G is degree-choosable. We shall characterize all graphs that degree-choosable, show that they are also degree AT. So degree AT and degree-choosable are equivalent. There is a slight difference between AW strict degree degenerate and degree-choosable.

Lemma 3.3.1 *Assume G is obtained from the disjoint union of two connected graphs G_1 and G_2, by identifying one vertex $v_1 \in V(G_1)$ and one vertex $v_2 \in V(G_2)$. Then G is not degree-choosable (respectively, not degree AT, or not AW strict degree degenerate) if and only if each G_i is not degree-choosable (respectively, not degree AT, or not AW strict degree degenerate).*

Proof. Assume first that each G_i is not degree-choosable. Let L_i be a list assignment of G_i such that $|L_i(V)| = d_{G_i}(v)$ for $v \in V(G_i)$ and G_i is not L_i-colourable. By renaming the colours, if needed, we may assume that $L_1(v_1) \cap L_2(v_2) = \emptyset$. Let v^* be the identification of v_1 and v_2 in G. Let $L(v) = L_i(v)$ if $v \in V(G_i) - \{v_i\}$ and $L(v^*) = L_1(v_1) \cup L_2(v_2)$. Since $d_G(v^*) = d_{G_1}(v_1) + d_{G_2}(v_2)$, $|L(v)| = d_G(v)$ for each vertex v of G.

Now we show that G is not L-colourable. Assume to the contrary that ϕ is an

L-colouring of G. Then $\phi(v^*) \in L_i(v_i)$ for some $i \in \{1, 2\}$, and hence the restriction of ϕ to G_i is an L_i-colouring of G_i, a contradiction.

Next we assume that one of G_1, G_2 is degree-choosable, say G_2 is degree choosable. Assume L is a degree-assignment of G. We order the vertices of G_1 as $u_1, u_2, \ldots, u_n$ so that $u_n = v_1$ and each u_i has at least one neighbour u_j with $j > i$. We greedily colour the vertices of $G_1 - \{v_1\}$ in this order. As $L(u_i) = d_G(u_i)$ and u_i has at least one uncoloured neighbour, all the vertices of $G_1 - \{v_1\}$ can be coloured.

Let ϕ be the L_1-colouring of $G_1 - \{v_1\}$. Let $L'(v^*) = L(v^*) - \{\phi(u) : u \in N_{G_1}(v_1)\}$. Then $|L'(v)| \geq d_{G_2}(v)$ for each vertex v of G_2. Since G_2 is degree-choosable, G_2 has an L'-colouring ψ. The union $\phi \cup \psi$ is an L-colouring of G.

Assume G is AW strict degree degenerate, and (D, w) is an arc-weighted acyclic orientation of G with $d^+_{(D,w)}(v) < d_G(v)$ for all $v \in V(G)$. Let D_i be the restriction of D to G_i. Then for some $i \in \{1, 2\}$, $d^+_{(D_i,w)}(v) < d_{G_i}(v)$. Hence G_i is AW strict degree degenerate.

Conversely, assume G_1 or G_2, say G_2, is AW strict degree degenerate, and (D_2, w_2) is an arc-weighted acyclic orientation of G_2 with $d^+_{(D_2,w_2)}(v) < d_{G_2}(v)$ for all $v \in V(G_2)$. Let D_1 be an acyclic orientation of G_1 in which v_1 is the unique source. Let $D = D_1 \cup D_2$, and let $w(e) = w_2(e)$ if $e \in E(G_2)$ and $w(e) = 1$ if $e \in E(G_1)$. Then (D, w) is an arc-weighted acyclic orientation of G with $d^+_{(D,w)}(v) < d_G(v)$ for all $v \in V(G)$.

The conclusion on degree AT is proved similarly and omitted. ∎

By Lemma 3.3.1, to characterize degree-choosable graph (or degree AT graphs or AW strict degree degenerate graphs), it sufficces to consider the case that G is 2-connected.

As $\chi(K_n) = n = \Delta(K_n) + 1$ and $\chi(C_{2k+1}) = 3 = \Delta(C_{2k+1}) + 1$, we know that complete graphs and odd cycles are not degree-choosable, and hence not degree AT and not AW strict degree degenerate. Even cycles are degree AT and hence degree choosable, but not AW strict degree degenerate.

Definition 3.3.2 *We call a 2-connected graph G* trivial *if G is a complete graph or a cycle. A* minimal non-trivial 2-connected graph *is a non-trivial 2-connected graph G such that any proper induced subgraph of G is either not 2-connected or is a trivial 2-connected graph.*

Definition 3.3.3 *A* theta graph *is a graph consisting three paths connecting two vertices. If the three paths have lengths k_1, k_2, k_3, then it is denoted by θ_{k_1,k_2,k_3}.*

Recall that the length of a path is the number of edges in the path. Thus $\theta_{1,2,2}$ is the graph obtained from K_4 by deleting one edge (which is often denoted by K_4^-).

The following lemma is easy and its proof is left as an exercise.

Lemma 3.3.4 *If G is a minimal non-trivial 2-connected graph, then $G = \theta_{k_1,k_2,k_3}$ for some positive integers k_1, k_2, k_3.*

As G is a simple graph, we may assume that $k_2, k_3 \geq 2$.

The following lemma is also easy and its proof is left as an exercise.

Lemma 3.3.5 *If G is a minimal non-trivial 2-connected graph, then G contains an edge uv such that $d_G(u) = 3, d_G(v) = 2$ and there is a orientation D of D in which $(u, v) \in E(D)$ and $D - \{(u, v)\}$ is acyclic, with u be a sink and v be the unique source.*

Assume G is a minimal non-trivial 2-connected graph. Let D be the orientation of G as described in Lemma 3.3.5. Let w be the weight assignment such that $w(uv) = 2$ and $w(e) = 1$ for all other edges of G. Then (D, w) is arc-weighted acyclic, and $d^+_{(D,w)}(w) < d_G(w)$ for $w \in V(G)$. So G is AW strict degree degenerate. The following lemma says that the conclusion applies to all non-trivial 2-connected graphs.

Lemma 3.3.6 *If G is a non-trivial 2-connected graph, then G is AW strict degree degenerate.*

Proof. Assume G is a non-trivial 2-connected graph. Then G has an induced subgraph H which is minimal non-trivial 2-connected. Let (D', w') be the arc-weighted acyclic orientation of H with $d^+_{(D',w')}(w) < d_H(w)$ for all $w \in V(H)$.

Let D'' be an acyclic orientation of $G - E(H)$, in which vertices in $V(H)$ are sources, and no other vertex is a source. Let $D = D' \cup D''$, and let $w(e) = 1$ for all the edges of G, except that $w(uv) = 2$. It is easy to verify that (D, w) is an arc-weighted acyclic orientation of G with $d^+_{(D,w)}(w) < d_G(w)$ for each vertex w. Therefore G is AW strict degree degenerate, and hence degree AT, as well as degree choosable. ∎

A connected graph G is called a *Gallai tree* (respectively, a GDP-tree) if every block of G is either a complete graph or an odd cycle (respectively, every block of G is either a complete graph or a cycle).

The following corollary follows from the discussion above.

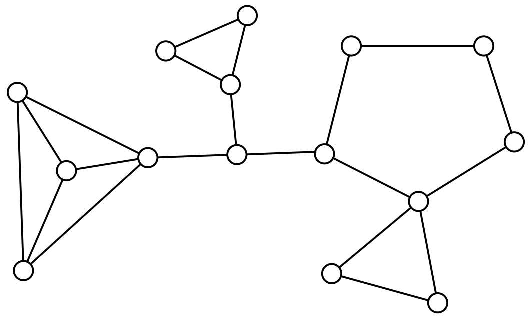

Figure 3.2: A Gallai tree

Corollary 3.3.7 *Assume G is a connected graph. Then the following are equivalent:*

1. *G is not degree choosable.*
2. *G is not degree AT.*
3. *G is a Gallai-tree.*

Moreover, G is not AW strict degree degenerate if and only if G is a GDP-tree.

3.4 Arc-weighted acyclic orientation of planar graphs

Definition 3.4.1 *A* planar graph *is a graph that can be* embedded in the plane*, i.e., it can be drawn on the Euclidean plane $\mathbb{R}^2$ in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.*

A drawing of a planar graph in the plane in which no two edges cross each other is called a *plane graph*, or a *planar embedding* of the graph. So a plane graph G is a subset of $\mathbb{R}^2$, each vertex v of G is a point in $\mathbb{R}^2$, and each edge $e = uv$ is a curve in $\mathbb{R}^2$ with end points u and v. These curves only intersects at their end points.

Definition 3.4.2 *Assume G is a plane graph. The connected components of $\mathbb{R}^2 - G$ are the* faces *of G.*

For a plane graph G, we denote by $V(G), E(G), F(G)$ the sets of vertices, edges and faces, respectively. The following *Euler formula* can be easily proved by induction on the number of edges.

Proposition 3.4.3 *Assume G is a connected plane graph. Then*

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

It follows from Euler formula that every simple planar graph G has a vertex of degree at most 5. I.e., G is strict 6-degenerate. Therefore

$$\chi(G) \leq ch(G) \leq AT(G) \leq 6.$$

The famous Four Colour Theorem says that every planar graph G has $\chi(G) \leq 4$. However, there are planar graphs that are not 4-choosable. The following example is modified from an example given by Choi and Kwon [7].

Theorem 3.4.4 *There exists a non-4-choosable planar graph.*

Proof. Let H be the graph, and L, the list asignment of H as given in Figure 3.3.

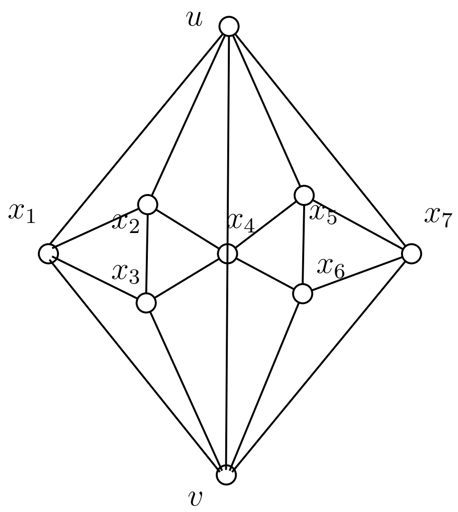

Figure 3.3: The graph H with list assignment L defined as $L(u) = \{1\}$, $L(v) = \{2\}$, $L(x_1) = \{1, 2, 3, 4\}$, $L(x_4) = \{1, 2, 4, 5\}$, $L(x_2) = \{1, 3, 4, 5\}$, $L(x_3) = \{2, 3, 4, 5\}$, $L(x_5) = \{1, 3, 4, 5\}$, $L(x_6) = \{2, 3, 4, 5\}$, $L(x_7) = \{1, 2, 3, 5\}$

First we show that H is not L-colorable. Assume to the contrary that ϕ is an L-coloring of H. Since $\phi(u) = 1$ and $\phi(v) = 2$, we conclude that $\phi(x_4) \in \{4, 5\}$. If $\phi(x_4) = 5$, then each of the three vertices in the triangle $\{x_1, x_2, x_3\}$ needs to be coloured by colors 3 and 4, a contradiction. If $\phi(x_4) = 4$, then each of the three vertices in the triangle $\{x_5, x_6, x_7\}$ needs to be coloured by colors 3 and 5, again a contradiction.

Now let G be the planar graph obtained from K_4 by adding, for every ordered pair of vertices (x, y) of K_4, a copy of H, identifying u with x and identifying v with y. Let $L(x) = \{1, 2, 3, 4\}$ for each vertex x of K_4, and for each copy of H and for each $x \in V(H) - \{u, v\}$, let $L(x)$ be as defined above. Then L is a 4-list assignment of G. Assume that ϕ is an L-coloring of G. Then one ordered pair of vertices of K_4 is coloured by 1 and 2. But such a coloring cannot be extended to an L-coloring of the copy of H associated with this pair of vertices. So G is not L-colorable. ∎

This non-4-choosable planar graph has 88 vertices. The smallest known non-4-choosable planar graph, due to Mirzakhani [26], has 63 vertices.

Since G is not 4-choosable, we have $AT(G) \geq 5$. The following theorem shows that $AT(G) \leq 5$ for every planar graph G.

Theorem 3.4.5 *[37] If G is a planar graph, then $ch(G) \leq AT(G) \leq st_{aw}(G) \leq 5$.*

Theorem 3.4.5 is a consequence of the following more technical result.

Assume G is a plane graph and v_1v_2 is a boundary edge of G. Let $f_{G,v_1v_2} \in \mathbb{N}^G$ be defined as follows:

$$f_{G,v_1v_2}(v) = \begin{cases} 1, & \text{if } v \in \{v_1, v_2\}, \\ 3, & \text{if } v \text{ is a boundary vertex and } v \notin \{v_1, v_2\}, \\ 5, & \text{otherwise.} \end{cases}$$

Theorem 3.4.6 *Assume that G is a plane graph and v_1v_2 is a boundary edge of G. Then $G - v_1v_2$ arc-weighgted strict f_{G,v_1v_2}-degenerate.*

Proof. We prove the theorem by induction on the number of vertices. If $|V(G)| \leq 3$, then this can be easily verified. Assume $|V(G)| = n \geq 4$, and the theorem holds for plane graphs with fewer vertices.

If G has a cut vertex w, then let G_1, G_2 be induced subgraphs of G with $V(G_1) \cap V(G_2) = \{w\}$ and $V(G_1) \cup V(G_2) = V(G)$. Assume that $v_1v_2 \in E(G_1)$. Let u be a boundary neighbour of w in G_2. By the minimality of G, $G_1 - v_1v_2$ has an arc-weighted acyclic orientation (D_1, w_1) such that $d^+_{(D_1,w_1)}(v) < f_{G_1,v_1v_2}(v)$ for each vertex v of G_1, and $G_2 - uw$ has an arc-weighted acyclic orientation (D_2, w_2) such that $d^+_{(D_2,w_2)}(v) < f_{G_2,uw}(v)$ for each vertex v of G_2.

Let D be obtained from $D_1 \cup D_2$ by adding the arc (u, v), and let $w(e) = w_i(e)$ if $e \in E(G_i) - \{uw\}$ and $w(uw) = 1$. Since $d^+_{(D_2,w_2)}(w) = 0$, we conclude that $d^+_{(D,w)}(v) < f_{G,v_1v_2}(v)$ for each vertex v of G.

Now we show that (D, w) is acyclic. Let D' be a non-empty subdigraph of D. If D' contains an edge of G_1, then let $D'_1 = D \cap D_1$. By the choice of (D_1, w_1), D'_1 contains a dominating arc e. Since $w(e) = w_1(e)$ and $d^+_{(D',w)}(v) = d^+_{(D'_1,w_1)}(v)$ for each vertex v of G_1, we conclude that e is a dominating arc of (D', w).

Assume D' contains no edge of G_1. If $(u, w) \in E(D')$, then since $d^+_{(D',w)}(w) = 0$, (u, w) is a dominating arc of (D', w). Otherwise, D' is a sub-digraph of D_2, and $d^+_{(D',w)}(v) = d^+_{(D',w_2)}(v)$ for each vertex of G_2. By our choice of (D_2, w_2), (D', w) has a dominating arc. Therefore (D, w) is indeed an arc-weighted acyclic digraph and $G - v_1v_2$ is arc-weighted strict f_{G,v_1v_2}-degenerate.

Thus we may assume that G is 2-connected and the boundary of G is a simple cycle $B_G = (v_1, v_2, \ldots, v_n)$.

If B_G has a chord v_iv_j, then let G_1, G_2 be the two induced subgraphs with $v_1v_2 \in G_1$, $V(G_1) \cap V(G_2) = \{v_i, v_j\}$ and $V(G_1) \cup V(G_2) = V(G))$. By the induction hypothesis, $G_1 - v_1v_2$ has an arc-weighted acyclic orientation (D_1, w_1) with $d^+_{(D_1,w_1)}(v) < f_{G_1,v_1v_2}(v)$ for each vertex v of G_1, and $G_2 - v_iv_j$ has an arc-weighted acyclic orientation (D_2, w_2) with $d^+_{(D_2,w_2)}(v) < f_{G_2,v_iv_j}(v)$ for each vertex v of G_2. Let $D = D_1 \cup D_2$ and $w(v) = w_1(v)$ if $v \in V(G_1)$, and $w(v) = w_2(v)$ if $v \in V(G_2) - \{v_i, v_j\}$. The same argument as above shows that (D, w) is an arc-weighted acyclic orientation of $G - v_1v_2$ with $d^+_{(D,w)}(v) < f_{G,v_1v_2}(v)$ for each vertex v of G.

Assume that G has no chord. Let $G' = G - v_n$. Let $v_1, u_1, u_2, \ldots, u_k, v_{n-1}$ be the neighbours of v_n. By induction hypothesis, $G' - v_1v_2$ has a an arc-weighted acyclic orientation (D', w') with $d^+_{(D',w')}(v) < f_{G',v_1v_2}(v)$ for each vertex of G'. Let D be obtained from D' by adding the arcs (v_n, v_1), (v_n, v_{n-1}) and (u_i, v_n) for $i = 1, 2, \ldots, k$. Let $w(e) = w'(e)$ for edges e of G' and $w(v_nv_1) = w(v_nv_{n-1}) = 1$ and $w(u_iv_n) = 2$ for $i = 1, 2, \ldots, k$.

It follows easily from the definition and induction hypothesis that $d^+_{(D,w)}(v) < f_{G,v_1v_2}(v)$ for each vertex v of G.

Now we show that (D, w) is acyclic. Let D'' be a non-empty sub-digraph of D. If $(v_n, v_1) \in E(D'')$, then (v_n, v_1) is a dominating arc, as $d^+_{(D'',w)}(v_1) = 0$.

Assume $(v_n, v_1) \notin E(D'')$. If $(u_i, v_n) \in E(D'')$ for some $i \in \{1, 2, \ldots, k\}$, then (u_i, v_n) is a dominating arc, as $w(u_i v_n) = 2$ and $d^+_{(D'',w)}(v_n) \le 1$.

Assume $(u_i, v_n) \notin E(D'')$ for all $i = 1, 2, \ldots, k$. If $E(D'') \cap E(D') \neq \emptyset$, then $(D' \cap D'', w')$ has a dominating arc, which is also a dominating arc of (D'', w) (as $d^+_{(D'',w)}(v) = d^+_{(D'' \cap D', w')}(v)$ for each vertex v of G'). Otherwise, (v_n, v_{n-1}) is a dominating arc. ∎

3.5 Defective list colouring of planar graphs

A *d-defective colouring* of a graph G is a colouring of the vertices of G such that each colour class induces a subgraph of maximum degree at most d. Thus, a 0-defective colouring of G is simply a proper colouring of G, while in a 1-defective colouring, a matching is allowed as a set of non-properly coloured edges. We say G is *d-defective k-colourable* if there is a d-defective colouring of G using a total number of k colours.

Given a k-list assignment L of G, a *d-defective L-colouring* of G is a d-defective colouring c of G with $c(v) \in L(v)$ for every vertex v of G. A graph G is *d-defective k-choosable* if for any k-list assignment L of G, there exists a d-defective L-colouring of G. Clearly, every d-defective k-choosable graph is d-defective k-colourable. The converse is not true.

It was proved by Cowen, Cowen and Woodall in [8] that every planar graph is 2-defective 3-colourable, and every outerplanar graph is 2-defective 2-colourable. Eaton and Hull [12], and Škrekovski [32] independently extended these results to list colouring, and showed that every planar graph is 2-defective 3-choosable, and every outerplanar graph is 2-defective 2-choosable. Cushing and Kierstead [9] proved that every planar graph is 1-defective 4-choosable. The result of Eaton and Hull and Škrekovski, and the result of Cushing and Kierstead can be formulated as follows:

Assume that G is a planar graph. For any 3-list assignment L of G, there is a subgraph H of G with $\Delta(H) \le 2$ such that $G - E(H)$ is L-colourable; For any 4-list assignment L of G, there is a subgraph H of G with $\Delta(H) \le 1$ such that $G - E(H)$ is L-colourable.

The subgraph H of G depends on the list assignment L. It is natural to ask whether one can choose a subgraph H of G independent of the list assignment L. In other words, we have the following questions:

Question 3.5.1 *Does every planar graph G have a subgraph H with $\Delta(H) \le 2$ such that $G \backslash E(H)$ is 3-choosable?*

Question 3.5.2 *Does every planar graph G have a subgraph H with $\Delta(H) \leq 1$ such that $G \backslash E(H)$ is 4-choosable?*

It turns out that the answer to Question 3.5.1 is negative, and the answer to Question 3.5.2 is positive.

A planar graph G for which $G \backslash E(H)$ is not 3-choosable for any subgraph H of G with $\Delta(H) \leq 3$ was given in [6]. In the next section, we present a postive answer to Question 3.5.2.

3.6 Planar graph minus a matching

Next we show that Question 3.5.2 has a positive answer.

Theorem 3.6.1 *Every planar graph G contains a matching M such that $AT(G \backslash M) \leq 4$.*

Theorem 3.6.1 was proved in [16], where the proof is a direct calculation of the coefficient of a certain monomial in the expansion of $f_G(\mathbf{x})$. Here we present a proof which is basically the same as the proof of Theorem 3.4.5, except that the technical statement is different.

Definition 3.6.2 *Assume that G is a plane graph, $e = v_1v_2$ is a boundary edge of G, and M is a matching in G containing e. An orientation D of $G \backslash M$ is* nice *for (G, e, M) if the following conditions hold:*

1. *D is an AT-orientation.*
2. *$d^+_D(v_1) = d^+_D(v_2) = 0$.*
3. *$d^+_D(v) \leq 2 - d_M(v)$ for every other boundary vertex v.*
4. *$d^+_D(v) \leq 3$ for each interior vertex v.*

Notice that $d_M(v) = 1$ if v is covered by M, and $d_M(v) = 0$ otherwise.

Theorem 3.6.1 follows from Theorem 3.6.3 given below.

Theorem 3.6.3 *Assume that G is a plane graph and $e = v_1v_2$ is a boundary edge of G. Then G has a matching M containing e such that there exists a nice orientation D for (G, e, M).*

Proof. Assume that the theorem is false and G is a minimum counterexample.

If G has a cut-vertex v, then let G_1, G_2 be two subgraphs of G separated by v, i.e., $V(G_1) \cap V(G_2) = \{v\}$ and $V(G_1) \cup V(G_2) = V(G)$. Assume $v_1v_2 \in E(G_1)$. Let $e' = vu$ be a boundary edge of G_2 incident to v. Then G_1 has a matching M_1 containing e such that there exists a nice orientation D_1 for (G_1, e, M_1), and Then G_2 has a matching M_2 containing e such that there exists a nice orientation D_2' for (G_2, e', M_2). Let $M = M_1 \cup (M_2) - \{e'\})$, D_2 be obtained from D_2' by adding arc (u, v). Then M is a matching in G, and by Lemma 2.2.13, D is a nice orientation for (G, e, M).

If $B(G)$ has a chord $e' = vu$, then let G_1, G_2 be two subgraphs of G separated by vu, i.e., $V(G_1) \cap V(G_2) = \{v, u\}$ and $V(G_1) \cup V(G_2) = V(G)$. Assume $v_1v_2 \in E(G_1)$. Then G_1 has a matching M_1 containing e such that there exists a nice orientation D_1 for (G_1, e, M_1), and Then G_2 has a matching M_2 containing e such that there exists a nice orientation D_2 for (G_2, e', M_2). Let $M = M_1 \cup (M_2) - \{e'\})$. Then M is a matching in G, and by Corollary 2.2.11, D is a nice orientation for (G, e, M).

Assume G has no cut-vertex and the boundary cycle B_G of G has no chord.

Let $G' = G - v_n$. Let $v_1, u_1, u_2, \ldots, u_k, v_{n-1}$ be the neighbours of v_n. By induction hypothesis, G' has a matching M' such that (G', e, M') has a nice orientation D'. Let D be obtained from D' by adding arcs (v_n, v_1) and (v_n, v_{n-1}) and arcs $a_i = (u_i, v_n)$ for $i = 1, 2, \ldots, k$. If D is an AT-orientation, then it is a nice orientation for (G, e, M') and we are done.

Assume that D is not an AT-orientation. For $i = 1, 2, \ldots, k$, $\mathcal{E}(D, a_i)$ is the set of Eulerian subdigraphs H of D which contains the arc $a_i = (u_i, v_n)$. Then

$$\mathcal{E}_e(D) = \mathcal{E}_e(D') \cup \bigcup_{i=1}^{k} \mathcal{E}_e(D, a_i), \text{ and } \mathcal{E}_o(D) = \mathcal{E}_o(D') \cup \bigcup_{i=1}^{k} \mathcal{E}_o(D, a_i).$$

Moreover the unions are disjoint unions. So

$$|\mathcal{E}_e(D)| = |\mathcal{E}_e(D')| + \sum_{i=1}^{k} |\mathcal{E}_e(D, a_i)|, \text{ and } |\mathcal{E}_o(D)| = |\mathcal{E}_o(D')| + \sum_{i=1}^{k} |\mathcal{E}_o(D, a_i)|.$$

Now $|\mathcal{E}_e(D)| = |\mathcal{E}_o(D)|$ and $|\mathcal{E}_e(D')| \neq |\mathcal{E}_o(D')|$ imply that $|\mathcal{E}_e(D, a_i)| \neq |\mathcal{E}_o(D, a_i)|$ for some $i \in \{1, 2, \ldots, k\}$.

Assume that $|\mathcal{E}_e(D, a_i)| \neq |\mathcal{E}_o(D, a_i)|$, where $i \in \{1, 2, \ldots, k\}$. Then D has a directed cycle C containing a_i. If $d_D^+(u_i) \leq 1$, then let D'' be obtained from $(D - C) \cup C^T$ by reversing the arc a_i^R (thus the edge u_iv_n is oriented as (u_i, v_n) in D''). The same argument as in the proof of Theorem 3.4.5 shows that D'' is an AT-orientation.

As $d^+_{D''}(u_i) = d^+_D(u_i) + 2 \leq 3$, D'' satisfies the out-degree conditions of being a nice orientation for (G, e, M'), and hence D'' is indeed a nice orientation for (G, e, M'). Otherwise $d^+_D(u_i) = 2$. As $d^+_D(u_i) \leq 2 - d_{M'}(u_i)$, we conclude that $d_{M'}(u_i) = 0$, i.e., u_i is not covered by M'. Let $M = M' \cup \{u_i v_n\}$. Then M is a matching in G containing e. Let D'' be obtained from $(D - C) \cup C^T$ by deleting the arc a_i^R. Then $d^+_{D''}(u_i) = d^+_D(u_i) + 1 = 3$, and hence D'' satisfies the out-degree conditions of being a nice orientation for (G, e, M). Again, the same argument as in the proof of Theorem 3.4.5 shows that D'' is an AT-orientation. Hence D'' is indeed a nice orientation for (G, e, M). ∎

Chapter 4

Interpolation formula

4.1 Coefficient of a highest degree monomial

The coefficient of a highest degree monomial in the expansion of a polynomial $f(x_1, x_2, \ldots, x_n)$, which is crucial when we invoke CNS, can be expressed by means of an interpolation of f at appropriate sets of points. This chapter explores applications of such interpolation formulas.

Assume that $f(\mathbf{x}) \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is a polynomial over $\mathbb{F}$, with degree $\deg f \leq d_1 + d_2 + \ldots + d_n$. Uwe Schauz [29] and independently Michał Lasoń [?] proved a strengthening of CNS, which shows that the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ in the expansion of f can be written as an interpolation of its values on any grid of size $(d_1, d_2, \ldots, d_n)$.

Let $\mathbf{S} := S_1 \times S_2 \times \cdots \times S_n$, where each S_i is a subset of $\mathbb{F}$. For $\mathbf{a} := (a_1, a_2, \ldots, a_n) \in \mathbf{S}$, let

$$N_{\mathbf{S}}(\mathbf{a}) = \prod_{i=1}^{n} \prod_{b \in S_i - \{a_i\}} (a_i - b).$$

We may think of $N_{\mathbf{S}}(\mathbf{a})$ as a normalizing factor w.r.t. $\mathbf{a}$. Let

$$\chi_{\mathbf{a}}(\mathbf{x}) = N_{\mathbf{S}}(\mathbf{a})^{-1} \prod_{i=1}^{n} \prod_{b \in S_i - \{a_i\}} (x_i - b).$$

Then $\chi_{\mathbf{a}}$ vanishes at all points of the grid $\mathbf{S}$, except that $\chi_{\mathbf{a}}(\mathbf{a}) = 1$.

Theorem 4.1.1 *Assume that $f(\mathbf{x}) \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is a polynomial over $\mathbb{F}$, $\mathbf{S} := S_1 \times S_2 \times \cdots \times S_n$ is an n-dimensional grid over $\mathbb{F}$, $t_i = |S_i| - 1$, $\mathbf{t} = (t_1, t_2, \ldots, t_n)$,* *and* $\deg f \le \sum_{i=1}^{n} t_i$. *Then the coefficient* $c_{f,\mathbf{t}}$ *of the monomial* $\prod_{i=1}^n x_i^{t_i}$ *in the expansion of* $f(\mathbf{x})$ *is given by*

$$c_{f,\mathbf{t}} = \sum_{\mathbf{a}\in\mathbf{S}} N_{\mathbf{S}}(\mathbf{a})^{-1} f(\mathbf{a}).$$

Proof. Let

$$p(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbf{S}} \chi_{\mathbf{a}}(\mathbf{x}) f(\mathbf{a}).$$

By definition, $p(\mathbf{x})$ is a polynomial of degree $\sum_{i=1}^n t_i$, and it is easy to see that $c_{\chi_{\mathbf{a}},\mathbf{t}} = N_{\mathbf{S}}(\mathbf{a})^{-1}$, and hence $c_{p,\mathbf{t}} = \sum_{\mathbf{a}\in\mathbf{S}} N_{\mathbf{S}}(\mathbf{a})^{-1} f(\mathbf{a})$. Since $\chi_{\mathbf{a}}(\mathbf{x}) = 1$ if $\mathbf{x} = \mathbf{a}$ and $\chi_{\mathbf{a}}(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathbf{S} - \{\mathbf{a}\}$, we have $p(\mathbf{a}) = f(\mathbf{a})$ for $\mathbf{a} \in \mathbf{S}$. Thus $f - p$ is a polynomial of degree at most $\sum_{i=1}^n t_i$, and $(f-p)(\mathbf{a}) = 0$ for all $\mathbf{a} \in \mathbf{S}$. As $|S_i| = t_i + 1$ for $i = 1, 2, \ldots, n$, by Theorem 1.2.1, the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in $f - p$ is $c_{f-p,\mathbf{t}} = c_{f,\mathbf{t}} - c_{p,\mathbf{t}} = 0$. Therefore

$$c_{f,\mathbf{t}} = c_{p,\mathbf{t}} = \sum_{\mathbf{a}\in\mathbf{S}} N_{\mathbf{S}}(\mathbf{a})^{-1} f(\mathbf{a}).$$

∎

4.2 Calculation of $N_{\mathbf{S}}(\mathbf{a})$

To apply Theorem 4.1.1, we need to calculate $N_{\mathbf{S}}(\mathbf{a})$ for all $\mathbf{a} \in \mathbf{S}$. This seems to be a nontrivial task. However, in some cases, we have simple formulas for $N_{\mathbf{S}}(\mathbf{a})$. Lemma 4.2.1 below, proved in [29], shows that for some special sets S, there is a simple formula for $\prod_{b\in S-\{a\}}(a-b)$.

For subset S of $\mathbb{F}$, let $f_S(x) = \prod_{b\in S}(x-b)$. Then $N_{\mathbf{S}}(\mathbf{a}) = \prod_{i=1}^n f_{S_i-\{a_i\}}(a_i)$. The following lemma gives formulas for $f_{S-\{a\}}(a)$ for some special sets S.

Lemma 4.2.1 *Assume that* $\mathbb{F}$ *is a field. For a positive integer* k*, let* $R_k = \{c \in \mathbb{F} : c^k = 1\}$ *be the set of* k*th roots of unity in the field* $\mathbb{F}$*. Assume that* $|R_k| = k$*.*

1. *If* $S = R_k$*, then for* $a \in S$*,* $f_{S-\{a\}}(a) = ka^{-1}$*. In particular, if* $R_k \cup \{0\}$ *is a finite subfield of* $\mathbb{F}$*, then* $f_{S-\{a\}}(a) = -a^{-1}$*.*

2. *If* $S = R_k \cup \{0\}$*, then for* $a \in S - \{0\}$*,* $f_{S-\{a\}}(a) = k1$*, and* $f_{S-\{0\}}(0) = -1$*.*

3. *If* S *is a subfield of* $\mathbb{F}$*, then for* $a \in S$*,* $f_{S-\{a\}}(a) = -1$*.*

4. *If $S=\{0,1,\ldots,d\}$, then for $a\in S$, $f_{S-\{a\}}(a)=(-1)^{d+a}d!\binom{d}{a}^{-1}$.*

Proof. (1) Since

$$f_{R_k}(x)=\prod_{b\in R_k}(x-b)=x^k-1=(x-1)(x^{k-1}+x^{k-2}+\ldots+x^0),$$

we conclude that $f_{R_k-\{1\}}(x)=f_{R_k}(x)/(x-1)=x^{k-1}+x^{k-2}+\ldots+x^0$ and $f_{R_k-\{1\}}(1)=k$.

For $a\in R_k$, $aR_k=\{ba:b\in R_k\}=R_k$ and $a(R_k-\{1\})=R_k-\{a\}$. Hence $f_{R_k-\{a\}}(a)=f_{a(R_k-\{1\})}(a)=\prod_{b\in R_k-\{1\}}(a-ab)=a^{k-1}f_{R_k-\{1\}}(1)=ka^{k-1}=ka^{-1}$. If $R_k\cup\{0\}$ is a finite subfield $\mathbb{F}_{p^t}$ of $\mathbb{F}$, then $|R_k|=k=p^t-1=-1\pmod{p^t}$, and hence $f_{R_k-\{a\}}(a)=-a^{-1}$. This proves (1).

(2) If $S=R_k\cup\{0\}$, then for $a\in S-\{0\}$, $f_{S-\{a\}}(a)=af_{R_k-\{a\}}(a)$. By (1) $f_{R_k-\{a\}}(a)=ka^{-1}$. Hence $f_{S-\{a\}}(a)=k$. As $f_{S-\{0\}}(x)=f_{R_k}(x)=x^k-1$, we have $f_{S-\{0\}}(0)=-1$.

(3) If S is a finite subfield $\mathbb{F}_{p^t}$ of $\mathbb{F}$, then $S=R_k\cup\{0\}$ for $k=p^t-1=-1$ and the conclusion follows from (2).

(4) Plug in the values in the definition of $f_{S-\{a\}}(a)$, we have

$$f_{S-\{a\}}(a)=\left(\prod_{0\le b<a}(a-b)\right)\prod_{a<b\le d}(a-b)=a!(d-a)!(-1)^{d-a}=(-1)^{d+a}d!\binom{d}{a}^{-1}.$$

∎

Corollary 4.2.2 *Assume that $\mathcal{F}$ is a finite field of order q, $f(\mathbf{x})\in\mathcal{F}[x_1,x_2,\ldots,x_n]$ and $\deg f\le n(q-1)$. Let $\mathbf{t}=(q-1,q-1,\ldots,q-1)$. Then*

$$c_{f,\mathbf{t}}=(-1)^n\sum_{\mathbf{a}\in\mathcal{F}^n}f(\mathbf{a}).$$

Proof. By (3) of Lemma 4.2.1, for $\mathbf{S}=\mathcal{F}^n$, we have $N_{\mathbf{s}}(\mathbf{a})=(-1)^n$ for every $\mathbf{a}\in\mathbf{S}$. ∎

Corollary 4.2.3 *Assume that $f(\mathbf{x})\in\mathbb{F}[x_1,x_2,\ldots,x_n]$ is a polynomial over $\mathbb{F}$, and $d_i\ge 0$ are integers such that $\deg f\le\sum_{i=1}^n d_i$. Let $\mathbf{d}=(d_1,d_2,\ldots,d_n)$. Then the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ in the expansion of f is*

$$c_{f,\mathbf{d}}=\left(\prod_{i=1}^n d_i!\right)^{-1}\sum_{a_1=0}^{d_1}\cdots\sum_{a_n=0}^{d_n}(-1)^{d_1+a_1}\binom{d_1}{a_1}\cdots(-1)^{d_n+a_n}\binom{d_n}{a_n}f(a_1,\ldots,a_n).$$

In particular, if $d_i = d$ *for all* i*, then the coefficient of the monomial* $\prod_{i=1}^n x_i^d$ *in the expansion of* f *is*

$$c_{f,\mathbf{d}} = (d!)^{-n} \sum_{\sigma} \left(\prod_{i=1}^n (-1)^{d+\sigma(x_i)} \binom{d}{\sigma(x_i)} \right) f(\sigma),$$

where the summation is over all mappings $\sigma : \{x_1, x_2, \ldots, x_n\} \to \{0, 1, \ldots, d\}$ *and* $f(\sigma)$ *is the evaluation of* f *at* $\sigma(x_i)$ *for* $i = 1, 2, \ldots, n$.

Proof. For $S_i = \{0, 1, \ldots, d_i\}$, $1 \le i \le n$, and $\mathbf{S} = S_1 \times S_2 \times \ldots \times S_n$, it follows from (4) of Lemma 4.2.1 that $N_{\mathbf{S}}(\mathbf{a}) = \prod_{i=1}^n \frac{1}{d_i!}(-1)^{d_i - a_i}\binom{d_i}{a_i}$ for $\mathbf{a} \in \mathbf{S}$. ∎

Corollary 4.2.3 is a classical result, proved before Theorem 4.1.1. Below is a simple direct proof of this corollary given in [30].

An alternative proof of Corollary 4.2.3

Let $S = \sum_{a_1=0}^{s_1} \cdots \sum_{a_n=0}^{s_n} (-1)^{s_1+a_1}\binom{s_1}{a_1} \cdots (-1)^{s_n+a_n}\binom{s_n}{a_n} f(a_1, \ldots, a_n)$. By linearity, it suffices to show that equality holds when $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n x_i^{t_i}$ is a monomial. Assume that $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n x_i^{t_i}$. Then

$$c_{f,\mathbf{s}} = \begin{cases} 1, & \text{if } \mathbf{s} = \mathbf{t}, \\ 0, & \text{otherwise.} \end{cases}$$

So we need to show that $S = \prod_{i=1}^n s_i!$ if $\mathbf{s} = \mathbf{t}$ and $S = 0$ if $\mathbf{s} \neq \mathbf{t}$. Plugging in the formula $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n x_i^{t_i}$ into the definition of S, we have

$$\begin{aligned} S &= \sum_{a_1=0}^{s_1} \cdots \sum_{a_n=0}^{s_n} (-1)^{s_1+a_1}\binom{s_1}{a_1} \cdots (-1)^{s_n+a_n}\binom{s_n}{a_n} \prod_{i=1}^n a_i^{t_i} \\ &= \prod_{i=1}^n \left(\sum_{j=0}^{s_i} (-1)^{s_i+j}\binom{s_i}{j} j^{t_i} \right). \end{aligned}$$

By inclusion-exclusion principle, $\sum_{j=0}^{s_i}(-1)^{s_i+j}\binom{s_i}{j} j^{t_i}$ counts the number of onto mappings from a t_i-set to an s_i-set. If $s_i = t_i$ for all i, then $S = \prod_{i=1}^n s_i!$. Otherwise, $t_i < s_i$ for some i, and $S = 0$. ∎

4.3 Alon-Tarsi number of $K_{2\star n}$ and cycle powers

In this section, we use Corollary 4.2.3 to calculate the coefficient of some monomials in graph polynomial.

We have sketched a proof of the result that $AT(K_{2\star n}) = n$ in Chapter 2 (Theorem **??**). In this section, we present a full proof of this theorem using the interpolation formula, which first appeared in [17].

An alternative proof of Theorem ??: Let $G = K_{2\star n}$. Let $V_1, V_2, \ldots V_n$ be the partite sets of G, with $V_i = \{u_i, v_i\}$. Since $\deg f_G = 2n(n-1)$, to prove Theorem **??**, it suffices to show that the monomial $\prod_{v \in V(G)} x_v^{n-1}$, which is of degree $2n(n-1)$, is non-vanishing. By Corollary 4.2.3, this is equivalent to showing that

$$\sum_{\sigma} \prod_{v \in V(G)} (-1)^{n-1+\sigma(v)} \binom{n-1}{\sigma(v)} f_G(\sigma) \neq 0$$

where the summation is over all mappings $\sigma : V(G) \to \{0, 1, \ldots, n-1\}$. However, if σ is not a proper coloring of G, then $f_G(\sigma) = 0$. Hence we may restrict the summation to proper colorings σ of G with color set $\{0, 1, \ldots, n-1\}$. As G is uniquely n-colorable, all the proper colorings σ of G are of the form $\sigma_\pi(u_i) = \sigma_\pi(v_i) = \pi(i)$, where π is a permutation of $\{0, 1, \ldots, n-1\}$. For each $i \in \{0, 1, \ldots, n-1\}$, there are exactly two vertices of color i. So we conclude that

$$\prod_{v \in V(G)} (-1)^{n-1+\sigma(v)} \binom{n-1}{\sigma(v)} = \prod_{i=0}^{n-1} \binom{n-1}{i}^2.$$

Also, if π' is obtained from π by interchanging colors i, j, then $f_G(\sigma_{\pi'})$ is obtained from $f_G(\sigma_\pi)$ by changing $(i-j)^4$ to $(j-i)^4$. Hence $f_G(\sigma)$ is a nonzero constant (it does not depend on σ). Therefore $\sum_\sigma \prod_{v \in V(G)} (-1)^{n-1+\sigma(v)} \binom{n-1}{\sigma(v)} f_G(\sigma) \neq 0$.

This completes the proof of Theorem **??**.

Given a graph G and a positive integer k, the power G^k is the graph with vertex set $V(G)$ in which two distinct vertices u and v are adjacent if and only if $d_G(u, v) \leq k$. Disproving a conjecture of Kostochka and Woodall [24], Kim and Park [22, 21] proved that there are graphs G for which $\chi(G^2) < ch(G^2)$. Furthermore, Kosar, Petrickova, Reiniger and Yeager [23] and Kim, Kwon and Park [20] proved independently that for any positive integer k, there is a graph G with $\chi(G^k) < ch(G^k)$. Nevertheless, if G is a cycle, then for any k, the Alon-Tarsi number of G^k equals its chromatic number and hence $\chi(G^k) = ch(G^k)$.

Theorem 4.3.1 *For any cycle C_n and for any positive integer k, $AT(C_n^k) = \chi(C_n^k)$.*

First we calculate the chromatic number of C_n^k. It turns out that it is easier to determine its circular chromatic number.

A (k,d)-*coloring* of a graph G is a mapping $c : V(G) \to \{0,1,\dots,k-1\}$ such that for any edge uv of G, $d \le |c(u)-c(v)| \le k-d$. The *circular chromatic number* of G is $\chi_c(G) = \min\{k/d : G \text{ has a } (k,d)\text{-coloring}\}$ [38]. As a $(k,1)$-coloring of G is simply a k-coloring of G, we have $\chi_c(G) \le \chi(G)$. On the other hand, for any (k,d)-coloring c of G, the mapping $\phi(v) = \lfloor c(v)/d \rfloor$ is a $\lceil \frac{k}{d} \rceil$-coloring of G. So for any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$.

Assume that c is a (k,d)-coloring of G. For $i = 0,1,\dots,k-1$, let $X_i = \{v : c(v) \in \{i, i+1, \dots, i+d-1\}\}$, here the summations are taken modulo k. Then each X_i is an independent set of G and hence $|X_i| \le \alpha(G)$. On the other hand, each vertex of v is contained in d of the sets X_i. So

$$d|V(G)| = \sum_{i=0}^{k-1} |X_i| \le k\alpha(G).$$

Hence $k/d \ge |V(G)|/\alpha(G)$.

Suppose now $G = C_n^k$ with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, in which $v_i v_j$ is an edge if $|i-j| \le k$ or $|i-j| \ge n-k$. Assume that $n = q(k+1)+p$, where $0 \le p \le k$.

Every set of $k+1$ consecutive vertices of C_n^k induces a complete graph K_{k+1} and hence contains at most one vertex of an independent set. Therefore G has independence number q and hence $\chi_c(G) \ge n/q$. On the other hand, if $c(v_i) = iq \pmod n$, then c is an (n,q)-coloring of G. Indeed, if $v_i \sim v_j$ and $i<j$, then $j-i \le k$ or $j-i \ge n-k$. Hence $q \le |c(v_j)-c(v_i)| \le n-q$. Thus we have $\chi_c(G) = n/q$. So if $n = q(k+1)+p$ where $0 \le p \le k$, then

$$\chi(G) = \lceil n/q \rceil = k+1+\lceil p/q \rceil.$$

We shall now prove that G has $AT(G) = \chi(G)$. The proof is by induction on $\lceil p/q \rceil$.

Case 1 $\lceil p/q \rceil = 0$.

In this case, $\chi(G) = n/q = k+1$ and we need to prove that $AT(G) = k+1$. If $q=1$, then $G = K_{k+1}$ and $k+1 = \chi(G) \le AT(G) \le \mathrm{col}(G) = k+1$. So assume that $q \ge 2$.

Note that f_G has degree $|E(G)| = kn$. So we need to show that the monomial $\prod_{i=1}^n x_i^k$ is non-vanishing. Let c be the coefficient of $\prod_{i=1}^n x_i^k$ in the expansion of f_G.

It follows from Corollary 4.2.3 that

$$(k!)^n c = \sum_{\sigma} \left((-1)^{\sum_{i=1}^n (k+\sigma(i))} \prod_{i=1}^n \binom{k}{\sigma(i)} \right) f_G(\sigma),$$

where the sum runs over all the mappings $\sigma : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k\}$. Note that if σ is not a proper coloring of G, then $f_G(\sigma) = 0$. Therefore, we can restrict the sum to those σ which are proper vertex colorings of G with colors $0, 1, \ldots, k$.

The graph G is uniquely $(k+1)$-colorable. All the $(k+1)$-colorings of G are obtained from a single $(k+1)$-coloring of G by permutations of colors. For $i \in \{1, 2, \ldots, n\}$, let $\{i\}_{k+1}$ be the unique integer in $\{0, 1, \ldots, k\}$ congruent to i modulo $(k+1)$. Given a permutation τ of $\{0, 1, \ldots, k\}$, let $\sigma_\tau : V(G) \to \{0, 1, \ldots, k\}$ be defined as follows:

$$\sigma_\tau(v_i) = \tau(\{i\}_{k+1}).$$

A mapping $\sigma : V(G) \to \{0, 1, \ldots, k\}$ is a proper coloring of G if and only if $\sigma = \sigma_\tau$ for some permutation τ of $\{0, 1, \ldots, k\}$. Therefore,

$$(k!)^n c = \sum_{\tau \in S_{k+1}} \left((-1)^{\sum_{i=1}^{n}(k+\sigma_\tau(v_i))} \prod_{i=1}^{n} \binom{k}{\sigma_\tau(v_i)} \right) f_G(\sigma_\tau),$$

where S_{k+1} is the set of all permutations of $\{0, 1, \ldots, k\}$.

For $\tau \in S_{k+1}$, for each $j \in \{0, 1, \ldots, k\}$, there are exactly q indices $i \in \{1, 2, \ldots, n\}$ with $\sigma_\tau(i) = j$. So

$$\begin{aligned} (-1)^{\sum_{i=1}^{n}(k+\sigma_\tau(i))} &= (-1)^{\frac{qk(k+1)}{2}}, \text{ and} \\ \prod_{i=1}^{n} \binom{k}{\sigma_\tau(i)} &= \prod_{i=0}^{k} \binom{k}{i}^q. \end{aligned}$$

Therefore $(-1)^{\sum_{i=1}^{n}(k+\sigma_\tau(v_i))} \prod_{i=1}^{n} \binom{k}{\sigma_\tau(v_i)}$ is a nonzero constant and $c \neq 0$ if and only if $\sum_{\tau \in S_{k+1}} f_G(\sigma_\tau) \neq 0$.

We shall show that $f_G(\sigma_\tau)$ is also a constant independent of τ. We view the cycle C_n as embedded in the plane as a circle, so that the vertices $v_1, v_2, \ldots, v_n$ of G are ordered cyclically. Each vertex v of G has k neighbours to its left and k neighbours to its right. Assume that D is the orientation of G in which each edge v_iv_j is oriented from v_i to v_j if v_i is to the left of v_j in the cyclic order. Assume that $f_G(\mathbf{x}) = \prod_{(u,v)\in A(D)}(x_u - x_v)$. We consider each pair of colors $i > j$. Each vertex of color i is adjacent to two vertices of color j, one to its left, and one to its right. These two edges incident to this vertex of color i contributes $(i-j)(j-i) = -(i-j)^2$ to $f_G(\sigma_\tau)$. There are q vertices of color i. So the contribution to $f_G(\sigma_\tau)$ from the pair of colors $i > j$ is $(-(i-j)^2)^q$. This contribution is independent of τ. So $f_G(\sigma_\tau)$ is a constant independent of τ. Hence $\prod_{i=1}^{n} x_n^k$ is a non-vanishing monomial of f_G and $AT(G) \leq k+1$. This completes the proof of Case 1.

Case 2 $\lceil p/q \rceil = 1$.

In this case, G is a subgraph of $C^{k+1}_{q(k+2)}$. Indeed, let the mapping $\phi : \{1, 2, \ldots, q(k+1) + p\} \to \{1, 2, \ldots, q(k + 2)\}$ be defined as

$$\phi(i) = \begin{cases} i + \lfloor \frac{i}{k} \rfloor, & \text{if } i < (q - p)k \\ i + (q - p), & \text{otherwise.} \end{cases}$$

Then ϕ is an embedding of $G = C^k_{q(k+1)+p}$ into $C^{k+1}_{q(k+2)}$. By Case 1, $AT(C^{k+1}_{q(k+2)}) \leq k+2$. By Lemma 2.1.6, $AT(G) \leq k + 2$.

Case 3 $\lceil p/q \rceil \geq 2$.

In this case, $n = q(k + 1) + p = q(k + 2) + (p - q)$, where $1 \leq p - q \leq k - q$. Hence $\chi(C^{k+1}_n) = (k+2)+\lceil (p-q)/q \rceil = \chi(G)$. By induction hypothesis $AT(C^{k+1}_n) = \chi(C^{k+1}_n)$. Therefore $AT(G) = \chi(G)$. ∎

4.4 4-regular graphs

This section proves the following result:

Let G be a 4-regular graph, and $\psi, \phi : V(G) \to \{0, 1, 2\}$ are two proper 3-colorings of G. We say ϕ and ψ are equivalent if the color classes $\{\phi^{-1}(i) : i \in \{0, 1, 2\}\}$ and $\{\psi^{-1}(i) : i \in \{0, 1, 2\}\}$ are the same, and we say ϕ and ψ have the same colour-class sizes if the multi-sets $\{|\phi^{-1}(0)|, |\phi^{-1}(1)|, |\phi^{-1}(2)|\}$ and $\{|\psi^{-1}(0)|, |\psi^{-1}(1)|, |\psi^{-1}(2)|\}$ are the same.

Theorem 4.4.1 *If G is a 4-regular graph, has an odd number of non-equivalent 3-colorings, and all the 3-colorings of G have the same colour-class sizes. Then*

$$AT(G) = 3.$$

Proof.

Assume G is a 4-regular n-vertex graph. We apply (**??**) to P_G with

$$S_v = \{0, 1, 2\}, \qquad t_v = 2 \quad (v \in V(G)).$$

We obtain

$$c_{G,2} = \sum_{\phi: V(G) \to \{0,1,2\}} \frac{P_G(\phi)}{D(\phi)}, \tag{4.1}$$

where

$$D(\phi) = \prod_{v \in V(G)} \prod_{a \in \{0,1,2\} \setminus \{\phi(v)\}} (\phi(v) - a).$$

If ϕ is not a proper colouring, then $P_G(\phi) = 0$. Hence only proper 3-colourings contribute to (4.8).

For a proper colouring ϕ, let

$$V_i = \phi^{-1}(i), \qquad n_i = |V_i| \quad (i = 0, 1, 2).$$

Then

$$D(\phi) = (-1)^{n_1} 2^{n_0 + n_2}, \tag{4.2}$$

as a vertex coloured 0 contributes a factor of $(0 - 1)(0 - 2) = 2$, a vertex coloured 1 contributes a factor of $(1 - 0)(1 - 2) = -1$, and a vertex coloured 2 contributes a factor of $(2 - 0)(2 - 1) = 2$ to $D(\phi)$.

Let e_{ij} be the number of edges between $\phi^{-1}(i)$ and $\phi^{-1}(j)$. Then

$$e_{01} + e_{02} = 4n_0, \tag{4.3}$$

$$e_{01} + e_{12} = 4n_1, \tag{4.4}$$

$$e_{02} + e_{12} = 4n_2. \tag{4.5}$$

Adding (4.3) and (4.4), and then subtracting (4.5), yields

$$2e_{01} = 4n_0 + 4n_1 - 4n_2,$$

and hence

$$e_{01} = 2(n_0 + n_1 - n_2) = 2(n - 2n_2).$$

Similarly,

$$e_{02} = 2(n - 2n_1), \qquad e_{12} = 2(n - 2n_0).$$

In particular, all e_{ij} are even.

An edge $e = uv$ contributes a factor of 2 to $|P_G(\phi)|$ if e is an edge between $\phi^{-1}(0)$ and $\phi^{-1}(2)$, and every other edge contribites a factor of 1 to $|P_G(\phi)|$. Thus

$$|P_G(\phi)| = 2^{2(n-2n_1)}. \tag{4.6}$$

Let $\mathrm{sgn}(\phi) \in \{1, -1\}$ be the sign of $P_G(\phi)$, i.e.,

$$P_G(\phi) = \mathrm{sgn}(\phi)|P_G(\phi)| = \mathrm{sgn}(\phi)2^{2(n-2n_1)}.$$

Then
$$\frac{P_G(\phi)}{D(\phi)} = \operatorname{sgn}(\phi)(-1)^{n_1}2^{n-3n_1} = \operatorname{sgn}(\phi)2^n(-2^{-3})^{n_1}. \tag{4.7}$$

If ψ is obtained from ϕ by interchanging the colors 0 and 1, then $\operatorname{sgn}(\psi) = \operatorname{sgn}(\phi)(-1)^{e_{01}} = \operatorname{sgn}(\phi)$. Similarly, if ψ is obtained from ϕ by interchanging the colors 2 and 1, then $\operatorname{sgn}(\psi) = \operatorname{sgn}(\phi)(-1)^{e_{12}} = \operatorname{sgn}(\phi)$. As (01) and (12) generates all the permutations of $\{0, 1, 2\}$, we conclude that if $\phi \sim \psi$, then $\operatorname{sgn}(\phi) = \operatorname{sgn}(\phi)$. Let $[\phi]$ be the set of the 6 colorings of G that are equivalent to ϕ. For each $i \in \{0, 1, 2\}$, there are exactly two coloring $\psi \in [\phi]$ with $\psi^{-1}(1) = \phi^{-1}(i)$. Therefore

$$\sum_{\psi \in [\phi]} \frac{P_G(\psi)}{D(\psi)} = \operatorname{sgn}(\phi)2^{n+1}\sum_{i=0}^{2}(-2^{-3})^{n_i}.$$

By our assumption, all the 3-colorings of G have the same color-class sizes. Assume the color-class sizes are a, b, c. Let $B = 2^n((-2^{-3})^a + (-2^{-3})^b + (-2^{-3})^b)$. Then

$$c_{G,2} = B\sum_{\phi} \operatorname{sgn}(\phi), \tag{4.8}$$

where the summation is over all non-equivalent 3-colorings of G.

It is easy to verify that $B \neq 0$. By our assumption, there is an odd number of non-equivalent 3-colorings of G. Therefore $c_{G,2} \neq 0$. This completes the proof of Theorem 4.4.1. ∎

An alternative proof of Theorem 2.5.2:

Assume G is a 4-regular graph on $3n$ vertices that decomposes into a Hamilton cycle and a family of n pairwise vertex-disjoint triangles. If ϕ is a 3-colouring of G, then each colour class contains one vertex of each triangle, and hence has size n.

It is left as a (difficult) exercise to prove G has an odd number of non-equivalent 3-colourings. Then it follows from Theorem 4.4.1 that $AT(G) = 3$.

4.5 Alon-Tarsi numbers of toroidal grids

Let $n, k \geq 3$ be integers. The Cartesian product $C_n \square C_k$ of two cycles is called a *toroidal grid*. Note that $C_n \square C_k$ is a 4-regular graph. If both n, k are even, then $C_n \square C_k$ is a bipartite graph. It follows from Corollary 2.3.2 that $AT(C_n \square C_k) = 3$.

The case that at least one of n, m is odd is more difficult and is settled in [25].

Theorem 4.5.1 *Assume s, k are positive integers and $G = C_{2s+1} \Box C_k$. Then*

$$AT(G) = \begin{cases} 3, & \text{if } k \text{ is even,} \\ 4, & \text{if } k \text{ is odd.} \end{cases}$$

The proof of Theorem 4.5.1 uses the concept of anti-Hermitian matrix and basic properties of complex numbers and matrices.

Definition 4.5.2 *The* conjugation $\overline{z}$ *of a complex number $z = a + bi$ is defined as $\overline{z} = a - bi$, where a, b are real numbers. The* conjugate transpose $\overline{M}$ *of a complex matrix M is defined as $\overline{M}_{i,j} = \overline{M_{j,i}}$. (Here $\overline{M}_{i,j}$ is the (i, j)th entry of the matrix $\overline{M}$, and $\overline{M_{j,i}}$ is the conjugate of the complex number $M_{j,i}$.) A square complex matrix M is* Hermitian *if $\overline{M} = M$ and M is* anti-Hermitian *(or* skew Hermitian*) if $\overline{M} = -M$.*

The following lemma lists some basic properties of conjugate transposes. These properties can be easily verified from the definitions.

Lemma 4.5.3 *Let A, B be complex matrices, $\mathbf{x}$ a complex vector and z, z' complex numbers. Then the following hold:*

1. $\overline{\overline{A}} = A$, $\overline{\overline{z}} = z$.
2. $\overline{(A + B)} = \overline{A} + \overline{B}$, $\overline{(z + z')} = \overline{z} + \overline{z'}$.
3. $\overline{(zA)} = \overline{z}\overline{A}$, $\overline{(1/z)} = 1/\overline{z}$.
4. $\overline{(AB)} = \overline{B}\ \overline{A}$, $\overline{(zz')} = \overline{z}\ \overline{z'}$.

Note that when we write the sum of two matrices, they are assumed to be of the same sizes; when we write the product AB of two matrices, the number of columns in A is assumed to be equal to the number of rows in B. We view an n-dimensional complex vector $\mathbf{x}$ as an $n \times 1$ matrix. So $\overline{\mathbf{x}}$ is a $1 \times n$ matrix.

Definition 4.5.4 *Assume that $\mathbf{u}, \mathbf{v}$ are n-dimensional complex vectors. Then $\overline{\mathbf{u}}\mathbf{v}$ is called the* inner *product of $\mathbf{u}$ and $\mathbf{v}$. The* norm *of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$ and the norm of a complex vector $\mathbf{u}$ is $|\mathbf{u}| = \sqrt{\overline{\mathbf{u}}\mathbf{u}}$.*

Note that the inner product of two complex vectors is a complex number.

Recall that a complex number λ is *pure imaginary* if its real part is zero, which is equivalent to $\lambda = -\overline{\lambda}$.

Lemma 4.5.5 *If M is an anti-Hermitian matrix, then all its eigenvalues are pure imaginary.*

Proof. Assume λ is an eigenvalue of M and $\mathbf{x}$ is the corresponding eigenvector. Then $\overline{\mathbf{x}}M\mathbf{x} = \lambda|\mathbf{x}|^2$. Hence

$$\overline{\lambda}|\mathbf{x}|^2 = \overline{\overline{\mathbf{x}}M\mathbf{x}} = \mathbf{x}\overline{M}\mathbf{x} = -\overline{\mathbf{x}}M\mathbf{x} = -\lambda|\mathbf{x}|^2.$$

Thus $\lambda = -\overline{\lambda}$ and hence λ is pure imaginary. ∎

Note that the same proof shows that if M is a Hermitian matrix, then all its eigenvalues are real.

Lemma 4.5.6 *Let s, k be positive integers and $n = 2s + 1$. Let $C_n = (v_1, v_2, \ldots, v_n)$ be an n-cycle, and $C_k = (w_1, w_2, \ldots, w_k)$ be a k-cycle, and $G = C_n \Box C_k$. Let $N = nk = |V(G)|$. Let $A = \{1, w, w^2\}$, where $w = e^{2\pi i/3}$ is a cube root of 1. Let $\mathcal{U}$ be the set of all proper colourings of C_n using colours from A. Each element of $\mathcal{U}$ is a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where $u_i \in A$ is the colour assigned to vertex v_i ($V(C_n) = \{v_1, v_2, \ldots, v_n\}$). For $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$, let*

$$M_{\mathbf{u},\mathbf{u}'} = P_{C_n}(u_1, \ldots, u_n) \cdot \prod_{i=1}^{n} \frac{u_i - u_i'}{\prod_{b \in A - \{u_i\}}(u_i - b)}.$$

Let $M = [M_{u,u'}]$ be the matrix whose rows and columns are indexed by elements of $\mathcal{U}$. Then

$$c_{G,d+1} = tr M^k.$$

Proof. By the coefficient formula in Theorem 4.1.1,

$$c_{G,2} \quad = \quad \sum_{(a_1,\ldots,a_N) \in A^N} \frac{P_G(a_1, \ldots, a_N)}{\prod_{i=1}^{N} \prod_{b \in A - \{a_i\}}(a_i - b)}. \tag{4.9}$$

We may restrict the summation to those $(a_1, \ldots, a_N) \in A^N$ that are proper colourings of G (for otherwise $P_G(a_1, \ldots, a_N) = 0$). Each proper colouring $(a_1, \ldots, a_N)$ of G corresponds to a sequence $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k$ of proper colourings of C_n, where $\mathbf{u}^i$ is the colouring of the ith copy of C_n, i.e., colouring of the those vertices of $G = C_n \Box C_k$ whose second coordinate is w_i ($V(C_k) = \{w_1, w_2, \ldots, w_k\}$). Let $\mathbf{u}^i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$. We denote by $P_{C_n}(\mathbf{u}_i) = P_{C_n}(u_{i,1}, u_{i,2}, \ldots, u_{i,n})$. Then

$$P_G(a_1, \ldots, a_N) = \prod_{i=1}^{k} P_{C_n}(\mathbf{u}^i) \cdot \prod_{i=1}^{k} \prod_{j=1}^{n} (u_{i+1,j} - u_{i,j}).$$

In the summation above, $u_{k+1,j} = u_{1,j}$. Therefore

$$\frac{P_G(a_1,\ldots,a_N)}{\prod_{i=1}^{N}\prod_{b\in A-\{a_i\}}(a_i-b)} = M_{\mathbf{u}^1,\mathbf{u}^2}\cdot M_{\mathbf{u}^2,\mathbf{u}^3}\cdot\ldots\cdot M_{\mathbf{u}^k,\mathbf{u}^1}.$$

The summation of all these products is exactly the trace of the matrix M^k. ∎

We shall show that the trace of M^k is non-zero if k is even, and trace of M^k is zero if k is odd.

Lemma 4.5.7 *The matrix M is an anti-Hermitian matrix.*

Proof. In this proof, the indices are modulo $2s+1$. Let $\mathbf{u} = (u_1, u_2, \ldots, u_{2s+1})$ be a proper 3-colouring of C_{2s+1} with colour set $A = \{1, w, w^2\}$. Let u_i^* be the unique element in $A - \{u_i, u_{i-1}\}$. Then

$$\frac{u_i - u_{i-1}}{\prod_{b\in A-\{u_i\}}(u_i-b)} = \frac{1}{u_i - u_i^*}.$$

As $P_{C_{2s+1}}(u_1,\ldots,u_{2s+1}) = \prod_{i=1}^{2s+1}(u_i - u_{i-1})$, we conclude that

$$\begin{aligned} M_{\mathbf{u},\mathbf{u}'} &= P_{C_{2s+1}}(u_1,\ldots,u_{2s+1})\cdot\prod_{i=1}^{2s+1}\frac{u_i-u_i'}{\prod_{b\in A-\{u_i\}}(u_i-b)} \\ &= \prod_{i=1}^{2s+1}\frac{(u_i-u_i')(u_i-u_{i-1})}{\prod_{b\in A-\{u_i\}}(u_i-b)} \\ &= \prod_{i=1}^{2s+1}\frac{u_i-u_i'}{u_i-u_i^*}. \end{aligned}$$

We shall now show that

$$M_{\mathbf{u},\mathbf{u}'} = -\overline{M_{\mathbf{u}',\mathbf{u}}}. \tag{4.10}$$

If $u_i = u_i'$ for some i, then $M_{\mathbf{u},\mathbf{u}'} = 0$. So (4.10) holds. Assume therefore that $u_i \neq u_i'$ for all i. (4.10) is equivalent to the following:

$$\prod_{i=1}^{2s+1}\frac{(u_i-u_i')(\overline{u}_i-\overline{u}_i^*)}{|u_i-u_i^*|^2} = -\prod_{i=1}^{2s+1}\frac{(\overline{u}_i'-\overline{u}_i)(u_i'-u_i'^*)}{|u_i'-u_i'^*|^2}. \tag{4.11}$$

Since $|u_i'-u_i'^*|^2 = |u_i-u_i^*|^2 = 3$, and $\overline{z} = \frac{1}{z}$ for $z\in\{u_i, u_i', u_i^*\}$, (4.11) is equivalent to

$$\prod_{i=1}^{2s+1}(u_i-u_i')\frac{u_i^*-u_i}{u_iu_i^*} = -\prod_{i=1}^{2s+1}(u_i'-u_i'^*)\frac{u_i-u_i'}{u_iu_i'}. \tag{4.12}$$

Now (4.12) is equivalent to

$$\prod_{i=1}^{2s+1} \frac{u_i^* - u_i}{u_i^*} = -\prod_{i=1}^{2s+1} \frac{u_i' - u_i'^*}{u_i'}. \tag{4.13}$$

Let $\epsilon_i = \frac{u_i}{u_{i-1}}$ and $\delta_i = \frac{u_i'}{u_{i-1}'}$. Then

$$\epsilon_i, \delta_i \in \{w, w^2\},\ u_i^* = u_i\epsilon_i,\ u_i'^* = u_i'\delta_i.$$

Hence

$$\frac{u_i}{u_i^*} = \overline{\epsilon}_i,\ \frac{u_i'^*}{u_i'} = \delta_i.$$

We re-write (4.13) as

$$\prod_{i=1}^{2s+1}(1 - \overline{\epsilon}_i) = -\prod_{i=1}^{2s+1}(1 - \delta_i).$$

As

$$\prod_{i=1}^{2s+1} \epsilon_i = \prod_{i=1}^{2s+1} \delta_i = 1,$$

(4.13) is equivalent to

$$\prod_{i=1}^{2s+1}(1 - \delta_i) = -\prod_{i=1}^{2s+1} \epsilon_i(1 - \overline{\epsilon}_i) = \prod_{i=1}^{2s+1}(1 - \epsilon_i). \tag{4.14}$$

Note that $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. So $1 - w = \sqrt{3}iw^2$ and $1 - w^2 = -\sqrt{3}iw$. So for $\epsilon \in \{w, w^2\}$, $1 - \epsilon = \pm i\sqrt{3}\epsilon^2$ and the signs are different for w and w^2. Therefore

$$(1 - \delta_i) = \begin{cases} 1 - \epsilon_i, & \text{if } \delta_i = \epsilon_i, \\ -(1 - \epsilon_i), & \text{if } \delta_i \neq \epsilon_i. \end{cases}$$

Hence

$$\prod_{i=1}^{2s+1}(1 - \delta_i) = = (-1)^m \prod_{i=1}^{2s+1}(1 - \epsilon_i), \tag{4.15}$$

where $m = |\{i : \delta_i \neq \epsilon_i\}|$.

Colour i with white if $u_i = u_i'w$, and colour i with black if $u_i = u_i'w^2$. Then $\epsilon_i = \delta_i$ if i and $i - 1$ have the same colour, and $\epsilon_i \neq \delta_i$ if i and $i - 1$ have distinct colours.

As there is an even number of indices i for which i and $i-1$ are of distinct colours, so m is even. Therefore (4.14) holds and hence M is an anti-Hermitian matrix. ∎

Now we are ready to complete the proof of Theorem 4.5.1. Assume that $G = C_{2s+1} \square C_k$. Let M be the matrix defined as above. Note that the matrix M is a non-zero matrix. If k is even, then since the eigenvalues of M are all imaginary, the eigenvalues of M^k are all real numbers, and if $k \equiv 0 \pmod 4$, then all the eigenvalues are non-negative; if $k \equiv 2 \pmod 4$, then all the eigenvalues are non-positive. Since M is non-zero, M^k has at least one non-zero eigenvalue. Hence $trM^k \neq 0$ (recall that the trace of a square matrix is the summation of all its eigenvalues). By Lemma 4.5.5, $c_{G,2} \neq 0$ and hence $AT(G) = 3$. If k is odd, then the eigenvalues of M^k are also all pure imaginary. Thus trM^k is a pure imaginary number. On the other hand, $trM^k = c_{G,2}$ is a real number. Hence $trM^k = c_{G,2} = 0$. So $AT(G) \geq 4$. As observed above, $AT(G) \leq 4$. So equality $AT(G) = 4$ holds. ∎

Corollary 4.5.8 *Assume that $n, m \geq 2$ are integers. If nm is even, then $ch(T_{n,m}) = 3$. If nm is odd, then $3 \leq ch(T_{n,m}) \leq 4$.*

In [5] it was conjectured that $ch(T_{n,m}) = 3$ for all $n, m \geq 3$. It remains open whether the conjecture holds when both n, m are odd.

Chapter 5

Line graphs

5.1 List colouring of line graphs

Although there are graphs whose choice numbers are much larger than their chromatic numbers (cf: Theorem 2.1.5), there are many graphs whose choice numbers equal their chromatic numbers. Such graphs are called *chromatic choosable*. The results in the previous section show that $K_{2\star n}$ and cycle powers C_n^k are chromatic choosable. The problem as to which graphs are chromatic choosable has been studied a lot in the literature. One of the most well-known problems is the so called "list colouring conjecture", which was suggested independently by many researchers, including Vizing, Gupta, Allbertson, Collins and Tucker.

Conjecture 5.1.1 (List coloring Conjecture) *For every graph G (parallel edges are allowed), the list edge-chromatic number $\chi_l'(G)$ equals its edge chromatic number $\chi'(G)$.*

Conjecture 5.1.1 is equivalent to the following conjecture, which is the restriction of Conjetcure 5.1.1 to r-edge colourable r-regular graphs.

Conjecture 5.1.2 (List coloring Conjecture) *Every r-edge colourable r-regular graph G is r-edge choosable.*

Lemma 5.1.3 *Conjecture 5.1.1 and Conjecture 5.1.2 are equivalent.*

Proof. It is obvious that Conjecture 5.1.1 implies Conjecture 5.1.2. To see the converse implication, we assume that Conjecture 5.1.2 holds and G is an arbitrary

r-edge colourable graph G. Let H be obtained from two vertex disjoint copies of G, say G and G', where the vertex set of G' is $V' = \{v' : v \in V(G)\}$, by adding $r - d_G(v)$ parallel edges between v and v' for each vertex v of G. Then H is r-regular and r-edge colourable. Since Conjecture 5.1.2 is assumed to be true for all r-edge colourable r-regular graphs, H is r-edge choosable. Hence G is also r-edge choosable. As G is an arbitrary r-edge colourable graph, Conjecture 5.1.1 is true. ∎

Let G be an n-vertex r-regular r-edge colourable graph. Let $L(G)$ be the line graph of G. We are interested in the Alon-Tarsi number of $L(G)$. In particular, we are interested in the problem whether $L(G)$ has Alon-Tarsi number r. As a strengthening of Conjecture 5.1.2, one may wonder if it is true that for any r-edge colourable r-regular graph G, its line graph $L(G)$ has Alon-Tarsi number r. However, this is not true. For example, if $G = K_{3,3}$, then G is 3-edge colourable and 3-regular, but its line graph $L(G)$ has Alon-Tarsi number 4. Nevertheless, for some classes of r-edge colourable r-regular graphs G, $L(G)$ does have Alon-Tarsi number r. In the next two sections, we shall show in the next two sections the following hold:

(1) If G is an r-edge colourable r-regular planar graph, then $L(G)$ has Alon-Tarsi number r.

(2) If $G = K_{p+1}$ and p is an odd prime, then $L(G)$ has Alon-Tarsi number p.

Let D be an orientation of $L(G)$ and let

$$f_D(\mathbf{x}) = \prod_{(e,e')\in E(D)} (x_e - x_{e'})$$

be the graph polynomial of $L(G)$ associated with D. Note that for a different orientation D' of $L(G)$, $f_D(\mathbf{x}) = \pm f_{D'}(\mathbf{x})$. We are only interested in whether a certain monomial in f_D is non-vanishing or not. So the choice of the orientation D is irrelevant. However, for some of the proofs, we may choose appropriate orientations D to make the calculations easier.

The line graph $L(G)$ is $2(r-1)$-regular (if e, e' are parallel edges in G, then e and e' are connected by two parallel edges in $L(G)$). Thus the number of variables x_e is $|E(G)| = nr/2$, and the degree of f_D is $|E(L(G))| = nr(r-1)/2$. Thus the Alon-Tarsi number of $L(G)$ is r if and only if the monomial $\prod_{e\in E(G)} x_e^{r-1}$ is non-vanishing.

By Corollary 4.2.3, the coefficient of the monomial $\prod_{e\in E(G)} x_e^{r-1}$ in $f_D(\mathbf{x})$ is equal to

$$((r-1)!)^{-nr/2} \sum_{\phi:E(G)\to\{0,1,\dots,r-1\}} \left(\prod_{e\in E(G)} (-1)^{(r-1+\phi(e))} \binom{r-1}{\phi(e)} \right) f_D(\phi).$$

If $\phi : E(G) \to \{0, 1, \dots, r-1\}$ is not a proper edge coloring of G, then $f_D(\phi) = 0$. Thus the summation above can be restricted to proper edge colorings ϕ of G. On the other hand, if $\phi : E(G) \to \{0, 1, \ldots, r-1\}$ is a proper edge coloring of G, then $\prod_{e\in E(G)}(-1)^{(r-1+\phi(e))}\binom{r-1}{\phi(e)}$ is a constant, as there are exactly $n/2$ edges e for which $\phi(e) = i$ for each $i \in \{0, 1, \ldots, r-1\}$. Therefore, to prove that the coefficient of the monomial $\prod_{e\in E(G)} x_e^{r-1}$ in $f_D(\mathbf{x})$ is nonzero, it is equivalent to showing that $\sum_{\phi\in\mathcal{C}} f_D(\phi) \neq 0$, where $\mathcal{C}$ is the set of mappings $\phi : E(G) \to \{0, 1, \ldots, r-1\}$ that are proper edge colorings of G.

For a vertex v of G, let $E(v)$ be the set of edges of G incident to v, and denote by Q_v the r-clique of $L(G)$ induced by $E(v)$. Let $<$ be a fixed ordering of the edges of G. Let

$$f_{D,v}(\mathbf{x}) = \prod_{(e,e')\in E(D), e,e'\in E(Q_v)} (x_e - x_{e'}).$$

Then $f_D(\mathbf{x}) = \prod_{v\in V(G)} f_{D,v}(\mathbf{x})$. If ϕ is a proper edge coloring of G, then for any vertex v of G,

$$|f_{D,v}(\phi)| = \prod_{0\le i<j\le r-1} (j-i), \text{ and hence } |f_D(\phi)| = \left(\prod_{0\le i<j\le r-1} (j-i)\right)^n.$$

For $\phi \in \mathcal{C}$, let $sign_D(\phi) = \pm 1$ (respectively, $sign_{D,v}(\phi) = \pm 1$), depending on whether $f_D(\phi)$ (respectively, $f_{D,v}(\phi)$) is positive or negative. As $|f_D(\phi)|$ is a constant, to show that the coefficient of $\prod_{e\in E(G)} x_e^{r-1}$ in $f_D(\mathbf{x})$ is nonzero, it is equivalent to showing that $\sum_{\phi\in\mathcal{C}} sign_D(\phi) \neq 0$. So we have the following lemma.

Lemma 5.1.4 *Assume that G is an r-edge colourable r-regular graph and D is an orientation of $L(G)$. Then $L(G)$ has Alon-Tarsi number r if and only if $\sum_{\phi\in\mathcal{C}} sign_D(\phi) \neq 0$, where $\mathcal{C}$ is the set of proper r-edge colourings of G.*

5.2 r-regular planar graphs

In this section, we prove that if G is an r-edge colourable r-regular planar graph, then $L(G)$ has Alon-Tarsi number r. This result was proved by Elligham and Goddyn [13]. The proof presented below is an unpublished proof given by the first author.

By Lemma 5.1.4, it suffices to show that for a fixed orientation D of $L(G)$, $\sum_{\phi\in\mathcal{C}} sign_D(\phi) \neq 0$.

For this purpose, we prove a stronger result:

Lemma 5.2.1 *There is an orientation D of $L(G)$ such that for all $\phi \in \mathcal{C}$, $sign_D(\phi) = (-1)^{\binom{r-1}{2}n/2}$, a nonzero constant.*

Proof. Assume that D, D' are distinct orientations of $L(G)$. If D and D' differ in an even number of edges, then $f_{D'}(\mathbf{x}) = f_D(\mathbf{x})$. Otherwise $f_{D'}(\mathbf{x}) = -f_D(\mathbf{x})$.

Let M be a one-factor of G. We define an orientation of $L(G)$ as follows: For each vertex v of G, order the edges in $E(v)$ as $(e_1, e_2, \ldots, e_r)$, where $e_1 \in M$ and the edges occur clockwise in this order in a plane embedding of G. Let Q_v be the $r-$ clique of $L(G)$ induced by the r edges at the vertex v of G. We orient the edges in Q_v as $e_i \to e_j$ where $1 \le i < j \le r$. The union of the orientations of all the cliques $\{Q_v : v \in V(G)\}$ gives an orientation of $L(G)$, which we denote by D_M. Note that for $e = uv$, the index of e in $E(v)$ and in $E(u)$ may be distinct.

Claim 5.2.2 *If M and M' are both one-factors of G, then D_M and $D_{M'}$ differ in an even number of edges. Consequently $f_{D_M}(\mathbf{x}) = f_{D_{M'}}(\mathbf{x})$ and $sign_{D_M}(\phi) = sign_{D_{M'}}(\phi)$ for $\phi \in \mathcal{C}$.*

Proof. For each vertex v of G, let α_v be the number of edges in Q_v for which D_M and $D_{M'}$ orient differently. We need to show that $\sum_{v \in V(G)} \alpha_v \equiv 0 \pmod 2$.

Let $e_{M,v}$ and $e_{M',v}$ be the edges of M and M' incident to v, respectively. If $e_{M,v} = e_{M',v}$, then $\alpha_v = 0$. Otherwise, let $[e_{M,v}, e_{M',v})$ be the set of edges in $E(v)$ from $e_{M,v}$ to $e_{M',v}$ (include $e_{M,v}$ but exclude $e_{M',v}$) in clockwise direction, and define the set $[e_{M',v}, e_{M,v})$ of edges similarly. It is easy to see that D_M and $D_{M'}$ orient the edge ee' in Q_v differently if and only if one of e, e' lies in $[e_{M,v}, e_{M',v})$, and the other lies in $[e_{M',v}, e_{M,v})$. Thus $\alpha_v = |[e_{M,v}, e_{M',v})| \times |[e_{M',v}, e_{M,v})|$. If r is odd, then α_v is even for every v and hence $\sum_{v \in V(G)} \alpha_v \equiv 0 \pmod 2$.

Assume that r is even. Then for each vertex v, $\alpha_v \equiv |[e_{M,v}, e_{M',v})| \equiv |[e_{M',v}, e_{M,v})| \pmod 2$. Let $C = (v_1, v_2, \ldots, v_{2k})$ be an even cycle of $M \cup M'$. The number of edges in $E(v_i)$ contained in the interior of C is $|[e_{M,v}, e_{M',v})| - 1$ or $|[e_{M',v}, e_{M,v})| - 1$, and hence has the same parity as $\alpha_v - 1$. Since each vertex in the interior of C has even degree r, by handshaking lemma, the total number of edges in the interior with one end on cycle C is even. Thus $\sum_{i=1}^{2k} \alpha_{v_i} \equiv \sum_{i=1}^{2k} (\alpha_{v_i} - 1) \equiv 0 \pmod 2$. As $(M \cup M') - (M \cap M')$ is the disjoint union of even cycles, we conclude that $\sum_{i=1}^{2k} \alpha_{v_i} \equiv 0 \pmod 2$. ∎

We shall prove that if M is a one-factor of G, then for any $\phi \in \mathcal{C}$, $sign_{D_M}(\phi) = (-1)^{\binom{r-1}{2} n/2}$. By Claim 5.2.2, we may assume that $M = \phi^{-1}(r-1)$ consists of edges of color $r - 1$.

Let G' be obtained from G by contracting all edges of color $r - 1$ (and all the other edges colored as in G). For each color $i \in \{0, 1, \ldots, r-2\}$, the subgraph of G' induced by edges of color i induce a 2-factor. We say *monochromatic cycles of colors i and j cross at u* if the cyclic order of the colors of the i- j-colored edges incident to u is i, j, i, j. Denote by β_u the number of pairs of monochromatic cycles of G' that cross at u.

For $u \in V(G')$, let $e_u = x_u y_u$ be the edge of G whose contraction results in u. For each pair of colors $0 \leq i, j \leq r-2$, in each of $P_{D_M,x_u}(\phi)$ and $P_{D_M,y_u}(\phi)$, $(j-i)$ or $(i-j)$ occurs as a factor. If cycles of colors i and j cross at u, then either $(j-i)$ or $(i-j)$ occurs twice in the product $P_{D_M,x_u}(\phi)P_{D_M,y_u}(\phi)$. If cycles of colors i and j do not cross at u, then each of $(j-i)$ and $(i-j)$ occurs once in $P_{D_M,x_u}(\phi)P_{D_M,y_u}(\phi)$. Note that $\binom{r-1}{2} - \beta_u$ is the number of pairs i, j for which monochromatic cycles of colors i and j do not cross. Therefore,

$$sign_{D_M,x_u}(\phi) sign_{D_M,y_u}(\phi) = (-1)^{\binom{r-1}{2} - \beta_u}.$$

Since every pair of monochromatic cycles cross each other an even number of times, we have $\sum_{u \in V(G')} \beta_u = 0 \pmod 2$. Therefore,

$$sign_{D_M}(\phi) = \prod_{u \in V(G')} sign_{D_M,x_u}(\phi) sign_{D_M,y_u}(\phi) = (-1)^{\binom{r-1}{2} n/2}.$$

This completes the proof of Lemma 5.2.1, as well as Theorem 5.2.3. ∎

Corollary 5.2.3 *If G is an r-regular r-edge colourable planar graph, then $L(G)$ has Alon-Tarsi number r, and hence G is r-edge paintable as well as r-edge choosable.*

5.3 Multicircuits

Assume $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ is a sequence of positive integers. Let $C_n[\mu]$ be a multicircuit with $V(C_n[\mu]) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(C_n[\mu]) = X_0 \cup X_1 \cup \ldots \cup X_{n-1}$, where X_i is a set of μ_i parallel edges joining v_i and v_{i+1} for each $i \in \{0, \ldots, n-1\}$ with $v_n = v_0$. Let $m = \sum_{i=0}^{n-1} \mu_i$. Note that $C_n[\mu]$ is a series-parallel graph. It is not difficult to prove (see [?]) that

$$\chi(L(C_n(\mu))) = \chi'(C_n(\mu)) = \max\{\Delta(C_n(\mu)), \lceil m/\lfloor n/2 \rfloor \rceil.$$

Theorem 5.3.1 *If G is a multicircuit graph on n vertices, m edges and maximum degree Δ, then* $\mathrm{AT}'(G) = \max\{\Delta, \lceil m/\lfloor n/2 \rfloor \rceil\}$.

We order the vertices of $G = L(C_n[\mu])$ along the orientation of C_n, so that vertices in K_{μ_i} are consecutive for each $i \in [0, n-1]$. Assume the vertices of G are

$u_0, u_1, \ldots, u_{m-1}$ in this cyclic order. For $i = 0, 1, \ldots, n-1$, let $s_i = \mu_0 + \mu_1 + \ldots + \mu_{i-1}$, and $t_i = \mu_0 + \mu_1 + \ldots + \mu_i - 1$, and let $I_i = [s_i, t_i]$ (by convention, $s_0 = 0$). The set $V_i = \{u_j : j \in I_i\}$ are the vertices of K_{μ_i} in G that correspond the μ_i parallel edges between v_i and v_{i+1} in $C_n[\mu]$, and $V_i \cup V_{i+1}$ induces a clique in G for $i = 0, 1, \ldots, n-1$ (addition in the indices are modulo n).

Example 5.3.2 *Let $n = 5$ and $\mu = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) = (1, 4, 1, 3, 2)$. Then $m = \sum_{i=0}^{4} = 11 = |V(G)|$. Figure 5.1 is the cyclic vertex order of $G = L(C_5[\mu])$ and $V_0 = \{v_0\}$, $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_1 = \{v_5\}$, $V_3 = \{v_6, v_7, v_8\}$ and $V_4 = \{v_9, v_{10}\}$.*

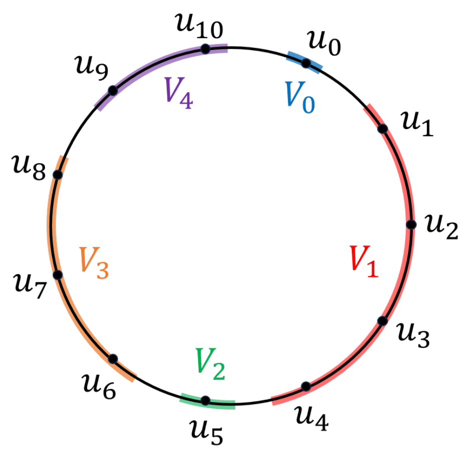

Figure 5.1: The cyclic vertex order of $G = L(C_5[\mu])$ with $\mu = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) = (1, 4, 1, 3, 2)$.

Let $C_m = u_0 u_1 \ldots u_{m-1} u_0$, which is a hamilton cycle in G. For $\alpha, \beta \in \{0, 1, \ldots, m-1\}$, $C_m[u_\alpha, u_\beta]$ denotes the subpath of C_m from u_α to u_β along the orientation given above. To be precise, if $\alpha < \beta$, then $C_m[u_\alpha, u_\beta] = u_\alpha u_{\alpha+1} u_{m-1}, \ldots u_\beta$; if $\alpha > \beta$, then $C_m[u_\alpha, u_\beta] = u_\alpha u_{\alpha+1} \ldots, u_{m-1}, u_0 \ldots u_\beta$.

Note that for $\alpha, \beta \in \{0, 1, \ldots, m-1\}$, $u_\alpha u_\beta$ is an edge of G if and only if there is an index i such that $\alpha, \beta \in I_i \cup I_{i+1}$ (the addition in indices are modulo m). Assume $u_\alpha u_\beta$ is an edge of G. Hence $C_m[u_\alpha, u_\beta] \subseteq C_m[s_i, t_{i+1}]$ for some i. The *length* of this edge with respect to C_m, denoted by $\ell_{C_m}(u_\alpha u_\beta)$, is defined to be the length of

$$\ell_{C_m}(u_\alpha u_\beta) = |C_m[u_\alpha, u_\beta]| - 1.$$

Note that the largest length of an edge of G is $\Delta - 1$, where $\Delta = \Delta(C_n[\mu]) = \max_{i \in [0, n-1]}(\mu_i + \mu_{i+1})$. Thus, G is a spanning subgraph of $C_m^{\Delta - 1}$. Furthermore, the following holds.

Lemma 5.3.3 *If $k \geq \Delta$ is a positive integer, and $m^\star \leq n\Delta - 2m$ is a positive integer, then*

$$G \subseteq C^{\Delta-1}_{m+m^\star} \subseteq C^{k-1}_{m+m^\star}.$$

Proof. Note that $\sum_{i=0}^{n-1}(\Delta - \mu_i - \mu_{i+1}) = n\Delta - 2\sum_{i=0}^{n-1}\mu_i = n\Delta - 2m$. Partition $m^* = \mu_1^* + \mu_2^* + \cdots + \mu_{n-1}^*$ with $0 \leq \mu_i^* \leq \Delta - (\mu_i + \mu_{i+1})$ for each $i \in [0, n-1]$.

For each $i \in [0, n-1]$, we insert $\mu_i^* \geq 0$ new vertices between V_i and V_{i+1}. We obtain a cycle $C_{m+m^\star}$. For an edge $u_\alpha u_\beta$ of G, with $\alpha, \beta \in I_i \cup I_{i+1}$, the length of the edge $u_\alpha u_\beta$ with respect to $C_{m+m^\star}$ is

$$\ell_{C_{m+m^\star}} \leq \ell_{C_m}(u_\alpha u_\beta) + u_i^* \leq \mu_i + \mu_{i+1} - 1 + u_i^* \leq \Delta - 1.$$

Hence, $u_\alpha u_\beta \in E(C^{\Delta-1}_{m+m^\star})$, and so G is a subgraph of $C^{\Delta-1}_{m+m^\star} \subseteq C^{k-1}_{m+m^\star}$. ∎

Example 5.3.4 *Let $n = 5$, $\mu = (1, 4, 1, 3, 2)$. Figure 5.1 illustrates the cycle C_m ($m = 11$) and the partition of its vertex set into $V_0, V_1, \ldots, V_4$. Note that*

$$\mu_2 + \mu_3 = 4,\ \mu_4 + \mu_0 = 3 \text{ and } \mu_i + \mu_{i+1} = 5 = \Delta \text{ otherwise.}$$

Let $\mu_2^\star = 1$ and $\mu_4^\star = 2$ and $\mu_i^\star = 0$ otherwise. Figure 5.2 illustrates the resulting cycle $C_{m'}$ after inserting $\sum_{i=1}^{4}\mu_i^* = 1 + 2 = 3$ new vertices. So $m' = m + \sum_{i=1}^{4}\mu_i^* = 11 + 1 + 2 = 14$.

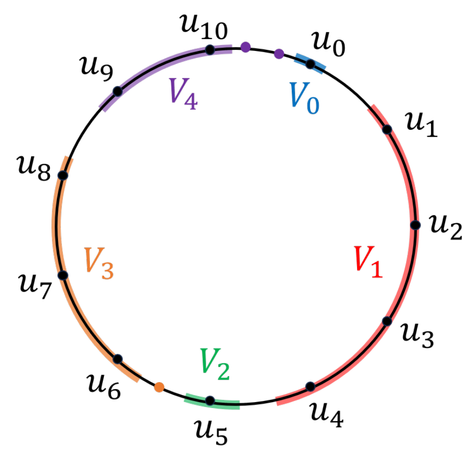

Figure 5.2: The example of $C_{m+m^\star}$ with $\mu_i^\star = 0$ except that $\mu_2^\star = 1$ (the new vertex is orange vertex) and $\mu_4^\star = 2$ (the new two vertices are purple vertices).

If $\Delta \geq \frac{m}{\lfloor n/2 \rfloor}$, then $\Delta \geq \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$ (as Δ is an integer), $\chi(G) = \Delta$ and $(n/2)\Delta \geq \lfloor n/2 \rfloor \Delta \geq m$. Let $m^\star = \lfloor n/2 \rfloor \Delta - m$. Then

$$n\Delta - 2m - m^\star = n\Delta - 2m - \lfloor n/2 \rfloor \Delta + m \geq (n/2)\Delta - m \geq 0.$$

It follows from Lemma 5.3.3 that $G \subseteq C^{\Delta-1}_{m+m^\star} = C^{\Delta-1}_{\lfloor n/2 \rfloor \Delta}$. Combining with Theorem 4.3.1, we have

$$\Delta = \chi(G) \leq AT(G) \leq AT(C^{\Delta-1}_{\lfloor n/2 \rfloor \Delta}) = \Delta.$$

Therefore, we have $\Delta = \chi(G) = AT(G)$.

Assume $\Delta < \frac{m}{\lfloor n/2 \rfloor}$. Let $t := \frac{m}{\lfloor n/2 \rfloor}$. Then $\chi(G) = \lceil t \rceil$ and $\lfloor t \rfloor \geq \Delta$. It follows from Lemma 5.3.3 that $G \subseteq C^{\Delta-1}_m \subseteq C^{\lfloor t \rfloor - 1}_m$ (by choosing $m^\star = 0$). Combining with Theorem 4.3.1, we have

$$\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \chi(G) \leq AT(G) \leq AT(C^{\lfloor t \rfloor - 1}_m) = \lceil \frac{m}{\lfloor \frac{m}{\lfloor t \rfloor} \rfloor} \rceil \leq \lceil \frac{m}{\lfloor \frac{m}{t} \rfloor} \rceil = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil.$$

This complete the proof of Theorem 5.3.1.

Chapter 6

Polynomials associated with matrices

6.1 Polynomials defined by matrices

Definition 6.1.1 *Let $A = [a_{i,j}]_{m\times n}$ be an $(m \times n)$ matrix, with $a_{i,j} \in \mathbb{F}$ for a field $\mathbb{F}$, and $b = (b_1, b_2, \dots, b_m) \in \mathbb{F}^m$. We define the polynomial $F_{A,b}(x_1, x_2, \dots, x_n) \in \mathbb{F}[x_1, x_2, \dots, x_n]$ as*

$$F_{A,b}(x_1, x_2, \dots, x_n) = \prod_{i=1}^{m} \sum_{j=1}^{n} (a_{i,j}x_j - b_i).$$

If $b_i = 0$ for $i = 1, 2, \dots, m$, then we write F_A for $F_{A,b}$.

Thus, polynomials $F_{A,b}$ (respectively, F_A) are polynomials (respectively, homogeneous polynomials) that can be factorized as the product of m linear factors. Therefore $\deg(F_{A,b}) = \deg(F_A) = m$.

We have presented three graph theory problems in Chapter 1.1 whose solutions correspond to non-zero points of polynomials in a given grid: List colouring of graphs, list avoiding orientation of graphs, and edge-weighting of graphs. Applications of Combinatorial Nullstellensatz to these three problems are the main objects of this lecture note. All the polynomials associated to these problems are of the form $F_{A,b}$ for some matrices.

Example 6.1.2 *For a graph G and an orientation D of G, let $A = [a_{e,v}]$ be the incidence matrix of D, defined as*

$$a_{e,v} = \begin{cases} 1, & \text{if } e = (v,u) \text{ for some } u, \\ -1, & \text{if } e = (u,v) \text{ for some } u, \\ 0, & \text{otherwise.} \end{cases}$$

Then the graph polynomial $P_D(x_v : v \in V(G)$ is equal to F_A.

Example 6.1.3 *Given a graph G and an orientation D of G, and a list $F'(v)$ of integers for each vertex v, in Chapter 1.1 we introduced the following polynomial*

$$g_{D,F'}(x_e : e \in E(G)) = \prod_{v \in V(G)} \prod_{b \in F'(v)} (\sum_{e \in E(G)} (a'_{e,v} x_e - b),$$

where $a'_{e,v}$ equals $a_{e,v}$ defined in Example 6.1.2.

Example 6.1.4 *Given a graph G and an orientation D of G, let $A = [a_{e,e'}]$ be the matrix whose rows and columns are indexed by edges of D, and for $e = (u,v)$, let*

$$A_{e,e'} = \begin{cases} 1, & \text{if } e' \in E_G(u) - \{e\}, \\ -1, & \text{if } e' \in E_G(v) - \{e\}, \\ 0, & \text{otherwise} \end{cases}$$

The h_D defined in Chapter 1.1 is defined as

$$h_D(x_e : e \in E) := \prod_{uv \in E(D)} (\sum_{e \in E(u)} x_e - \sum_{e \in E(v)} x_e).$$

It is easy to verify that $h_D = F_A$.

Both polynomials are defined by matrices. Given a graph G and an orientation D of G, we denote by A the matrix in Example 6.1.2, and by A' the matrix in Example 6.1.3. Note that the rows of A are indexed by edges in D, and columns of A are indexed by vertices in D. On the other hand, rows of A' are indexed by vertices of D and columns are indexed by edges of D. Indeed, A' is obtained from the transpose A^T of A by repeating, for each vertex v of D, $|F'(v)|$ times the row of A^T indexed by v.

Definition 6.1.5 *For a matrix $A = (a_{ij})_{1\le i\le n, 1\le j\le m}$ and two vectors $\alpha = (a_i)_{1\le i\le n} \in \mathbb{N}^n$, $\beta = (b_j)_{1\le j\le m} \in \mathbb{N}^m$ of non-negative integers, denote by $A^{\alpha,\beta}$ the matrix generated by copying a_i times the i-th row of A and b_j times the j-th column of A. Equivalently,*

$$A^{\alpha,\beta} = \begin{bmatrix} a_{11}J_{a_1\times b_1} & a_{12}J_{a_1\times b_2} & \cdots & a_{1m}J_{a_1\times b_m} \\ a_{21}J_{a_2\times b_1} & a_{22}J_{a_2\times b_2} & \cdots & a_{2m}J_{a_2\times b_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}J_{a_n\times b_1} & a_{n2}J_{a_n\times b_2} & \cdots & a_{nm}J_{a_n\times b_m} \end{bmatrix},$$

where $J_{a,b}$ is the all-1 $(a \times b)$ matrix.

We denote by 1 the all-1 vector, so $A^{1,1} = A$.

Definition 6.1.6 *For an $m \times m$ matrix $A = [a_{i,j}]$, the* permanent *of A is*

$$\mathrm{per}(A) = \sum_{\sigma\in S_m} \prod_{i=1}^{m} a_{i,\sigma(i)}.$$

Lemma 6.1.7 *Let $A = [a_{i,j}]$ be an $m \times n$ matrix, and $t = (t_1, t_2, \ldots, t_n)$ is a vector of non-negative integers with $\sum_{i=1}^n t_i = m$. Then*

$$c_{F_A,t} = \frac{1}{t!}\mathrm{per}(A^{1,\mathrm{t}}),$$

where $t! = \prod_{i+1}^n t_i!$.

Proof. To write F_A as a linear combination of monomials, for each $i \in [m]$, we pick one entry from the linear terms $\sum_{j=1}^n a_{i,j}x_j$, take the product to obtain a monomial, and then sum up all such monomials. Let Θ be the set of all mappings $\theta : [m] \to [n]$. Then

$$F_A = \sum_{\theta\in\Theta} \prod_{i=1}^{m} a_{i,\theta(i)} x_{\theta(i)}.$$

For a mapping $\theta \in \Theta$, the corresponding monomial is

$$\prod_{i=1}^{n} x_i^{|\theta^{-1}(i)|},$$

and the contribution of θ to the coefficient of this monomial is $\prod_{i=1}^m a_{i,\theta(i)}$.

Given $t = (t_1, t_2, \ldots, t_n)$ with $\sum_{i=1}^{n} t_i = m$, let

$$\Theta_t = \{\theta \in \Theta : |\theta^{-1}(i)| = t_i\}.$$

Then

$$c_{F_A,t} = \sum_{\theta \in \Theta_t} \prod_{i=1}^{m} a_{i,\theta(i)}.$$

Consider the matrix $A^{1,\mathrm{t}}$, which is a matrix that consists of t_i copies of the ith column of A for $i \in [n]$. Let C_i be the index of the t_i columns of $A^{1,\mathrm{t}}$ that are copies of the ith column of A. Let $A^{1,\mathrm{t}} = [a'_{l,j}]_{m\times m}$. Thus $a'_{l,j} = a_{i,j}$ if $l \in C_i$.

For each $\theta \in \Theta_t$, let Σ_θ be the subset of permutations σ of $[m]$ such that for each $i \in [n]$, if $j \in \theta^{-1}(i)$, then $\sigma(j) \in C_i$. Thus the restriction of σ to $\theta^{-1}(i)$ is a bijection from $\theta^{-1}(i)$ to C_i. For each $i \in [n]$, there are $t_i!$ bijections from $\theta^{-1}(i)$ to C_i. Thus $|\Sigma_\theta| = \prod_{i=1}^{n} t_i! = t!$. As $a'_{l,j} = a_{i,j}$ if $l \in C_i$, we have

$$\prod_{i=1}^{m} a_{i,\theta(i)} = \prod_{i=1}^{m} a'_{i,\sigma(i)}$$

for any $\sigma \in \Sigma_\theta$.

Note that S_m is the disjoint union of Σ_θ for $\theta \in \Theta_t$. Therefore

$$c_{F_A,t} = \sum_{\theta \in \Theta_t} \prod_{i=1}^{m} a_{i,\theta(i)} = \frac{1}{t!} \sum_{\sigma \in S_m} \prod_{i=1}^{m} a'_{i,\sigma(i)} = \frac{1}{t!}\mathrm{per}(A^{1,\mathrm{t}}).$$

∎

6.2 Trees with an even number of edges

There are many graphs that are not 2-edge-weight choosable. Nevertheless, experiment shows that most graphs are 2-edge-weight choosable. One question studied in [?] is which trees are 2-edge-weight choosable. An algorithm is presented in [?] that determines whether a given tree is 2-edge-weight choosable in linear time. The algorithm is a little complicated. However, Theorem 6.2.1 gives an easy sufficient condition for a tree to be 2-edge-weight choosable.

Theorem 6.2.1 *If T is a tree with an even number of edges, then* $\mathrm{per}(A_G) \neq 0$*, and consequently, T is 2-edge-weight choosable.*

In this section we prove that the parity of per$B(T)$ for a tree T is different from the parity of the number of edges of T. As a consequence, if T is a tree with an even number of edges, then per$B(T) \neq 0$ and so T is $(1,2)$-choosable by Theorem **??**.

Lemma 6.2.2 *Assume G has a cut edge e and G_1, G_2 are the two components of $G - e$. If G_i' is obtained from G_i by adding the edge e, then*

$$\mathrm{per}B(G) = \mathrm{per}B(G_1)\ \mathrm{per}B(G_2') + \mathrm{per}B(G_1')\ \mathrm{per}B(G_2).$$

Proof. Assume the edges of G_1 are $e_1, e_2, \ldots, e_{t-1}$, the cut edge $e = e_t$, and the edges of G_2 are $e_{t+1}, e_{t+2}, \ldots, e_m$. Then

$$B(G) = \begin{pmatrix} & \cdots & & * & & & \\ & B(G_1) & & * & & \mathbf{0} & \\ & \cdots & & * & & & \\ * & * & * & 0 & * & * & * \\ & & & * & & \cdots & \\ & \mathbf{0} & & * & & B(G_2) & \\ & & & * & & \cdots & \end{pmatrix}$$

where the row and the column containing $*$'s are row t and column t, and a $*$ represents an arbitrary real number. Let u^1 and u^2 be rows formed from row t of $B(G)$ by setting the first $t-1$ positions and the last $m-t$ positions to 0, respectively. Let B_i be obtained from $B(G)$ by replacing row t with u^i, i.e.,

$$B_1 = \begin{pmatrix} & \cdots & & * & & & \\ & B(G_1) & & * & & \mathbf{0} & \\ & \cdots & & * & & & \\ 0 & 0 & 0 & 0 & * & * & * \\ & & & * & & \cdots & \\ & \mathbf{0} & & * & & B(G_2) & \\ & & & * & & \cdots & \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} & \cdots & & * & & & \\ & B(G_1) & & * & & \mathbf{0} & \\ & \cdots & & * & & & \\ * & * & * & 0 & 0 & 0 & 0 \\ & & & * & & \cdots & \\ & \mathbf{0} & & * & & B(G_2) & \\ & & & * & & \cdots & \end{pmatrix}.$$

By linearity of permanent with respect to row vectors, we have

$$\mathrm{per}B(G) = \mathrm{per}B_1 + \mathrm{per}B_2.$$

The nonzero contribution to perB_1 from rows $t, t+1, \ldots, m$ are from columns $t, t+1, \ldots, m$. So perB_1 = per$B(G_1)$ per$B(G_2')$. Similarly we have perB_2 = per$B(G_1')$ per$B(G_2)$. Hence

$$\mathrm{per}B(G) = \mathrm{per}B(G_1)\ \mathrm{per}B(G_2') + \mathrm{per}B(G_1')\ \mathrm{per}B(G_2). \qquad \blacksquare$$

Lemma 6.2.3 *If T is a tree with m edges, then* $\mathrm{per}B(T) \equiv m-1 \pmod 2$.

Proof. Since we are only concerned about the modulo 2 value of the permanent, the calculations can be done in Z_2. Hence we can replace -1 by 1 in $B(T)$, and the resulting matrix is just the adjacency matrix of the line graph of G.

We prove the lemma by induction on m. If T is a star, then $B(T) = J_m - I_m$, where J_m is the $m \times m$ matrix with all the entries 1 and I_m is the $m \times m$ identity matrix. Hence $\mathrm{per}B(T)$ is the number of permutations of $\{1, 2, \ldots, m\}$ with no fixed point, which equals to the derangement number D_m. It is known that $D_1 = 0, D_2 = 1$ and $D_m = (m-1)(D_{m-1} + D_{m-2})$ for $m \geq 3$. Reducing modulo 2, by induction, we have $D_m \equiv m-1 \pmod 2$.

Assume that T is not a star. Let e be any edge of T so that each of the two components T_1 and T_2 of $T - e$ has at least one edge. By Lemma 6.2.2,

$$\mathrm{per}B(T) = \mathrm{per}B(T_1)\ \mathrm{per}B(T_2') + \mathrm{per}B(T_1')\ \mathrm{per}B(T_2),$$

where T_1, T_2, T_1', T_2' are as defined in Lemma 6.2.2. Assume T_1 has $t-1$ edges and T_2 has $m-t$ edges. By applying the induction hypothesis to each of T_1, T_1', T_2, T_2', we conclude that

$$\mathrm{per}B(T) \equiv (t-2)(m-t) + (t-1)(m-t-1) \equiv m-1 \pmod 2.$$

∎

Theorem 6.2.4 *If T is a tree with an even number of edges, then T is* $(1,2)$-*choosable.*

6.3 Every graph is $(2,3)$-choosable

In the same paper [35], Wong and Zhu raised the following conjectures:
Conjecture 1: There are constants k, k' such that every graph is (k, k')-choosable.
Conjecture 2: Every graph is (2,2)-choosable. Every graph with no isolated edges is $(1,3)$-choosable.

Moreover, they have constructed some special families of graphs which are $(2,2)$-choosable and some other families which are $(1,3)$-choosable. In answer to Conjecture 1, Wong and Zhu proved in a subsequent paper [36] the following theorem.

Theorem 6.3.1 *Every graph is* $(2,3)$-*choosable.*

Proof: The proof again uses CNS. For each $z \in V(G) \cup E(G)$, let x_z be a variable associated to z. Fix an arbitrary orientation D of G. We write $uv \in A(D)$ to mean that the edge uv is oriented from u to v. Consider the polynomial

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e=uv \in A(D)} \left(\left(\sum_{e \in E(v)} x_e + x_v \right) - \left(\sum_{e \in E(u)} x_e + x_u \right) \right).$$

P_G is a polynomial in $m+n$ variables x_a and is of degree m. Assign a real number $\phi(z)$ to the variable x_z, and view $\phi(z)$ as the weight of z. Let $P_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$. Then ϕ is a proper total weighting of G if and only if $P_G(\phi) \neq 0$. The question is under what condition one can find an assignment ϕ for which $P_G(\phi) \neq 0$.

An *index function* of G is a mapping η which assigns to each vertex or edge z of G a non-negative integer $\eta(z)$. An index function η of G is *valid* if $\sum_{z \in V \cup E} \eta(z) = |E|$. Note that $|E|$ is the degree of the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$. For a valid index function η, let c_η be the coefficient of the monomial $\prod_{z \in V \cup E} x_z^{\eta(z)}$ in the expansion of P_G. It follows from the CNS that if $c_\eta \neq 0$, and L is a list assignment which assigns to each $z \in V(G) \cup E(G)$ a set $L(z)$ of $\eta(z)+1$ real numbers, then there exists a mapping ϕ with $\phi(z) \in L(z)$ such that

$$P_G(\phi) \neq 0.$$

An index function η of G is called *non-singular* if there is a valid index function $\eta' \leq \eta$ (i.e., $\eta'(z) \leq \eta(z)$ for all $z \in V(G) \cup E(G)$) such that $c_{\eta'} \neq 0$. We now prove an important result.

Theorem 6.3.2 *Every graph G has a non-singular index function η with $\eta(v) \leq 1$ for $v \in V(G)$ and $\eta(e) \leq 2$ for $e \in E(G)$.*

As observed above, this implies that every graph G is $(2,3)$-choosable.

Let A_G be the matrix whose rows are indexed by edges in D and columns are indexed by edges and vertices of G, and for $e = uv \in A(D)$ and $z \in V(G) \cup E(G)$,

$$A_G[e,z] = \begin{cases} 1 & \text{if } z = v \text{, or } z \neq e \text{ is an edge incident to } v, \\ -1 & \text{if } z = u \text{, or } z \neq e \text{ is an edge incident to } u, \\ 0 & \text{otherwise.} \end{cases}$$

Then the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$ defined above can be written as

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e \in A(D)} \sum_{z \in V(G) \cup E(G)} A_G[e,z] x_z.$$

Given a vertex or edge z of G, let $A_G(z)$ be the column of A_G indexed by z. For an index function η of G, let $A_G(\eta)$ be the matrix with the property that each of its columns is a column of A_G, and each column $A_G(z)$ of A_G occurs $\eta(z)$ times as a column of $A_G(\eta)$. It is known [?] and easy to verify that for a valid index function η of G, $c_\eta \neq 0$ if and only if $\mathrm{per}(A_G(\eta)) \neq 0$, where $\mathrm{per}(A)$ denotes the permanent of the square matrix A. Thus a valid index function η of G is non-singular if and only if $\mathrm{per}(A_G(\eta)) \neq 0$. Recall that if A is an $m \times m$ matrix, then

$$\mathrm{per}(A) = \sum_{\sigma \in S_m} A[i, \sigma(i)],$$

where S_m is the symmetric group of degree m.

It is well-known (and follows easily from the definition) that the permanent of a matrix is multi-linear on its column vectors (as well as its row vectors): If a column C of A is a linear combination of two column vectors $C = \alpha C' + \beta C''$, and A' (respectively, A'') is obtained from A by replacing the column C with C' (respectively, with C''), then

$$\mathrm{per}(A) = \alpha \mathrm{per}(A') + \beta \mathrm{per}(A''). \tag{1}$$

Assume A is a square matrix whose columns are expressed as linear combinations of columns of A_G. Define an index function $\eta_A : V(G) \cup E(G) \to \{0, 1, \dots, \}$ as follows:

For $z \in V(G) \cup E(G)$, $\eta_A(z)$ is the number of columns of A in which $A_G(z)$ appears with nonzero coefficient.

The column vectors of A_G are not linearly independent. As observed in [?], for an edge $e = uv$ of G,

$$A_G(e) = A_G(u) + A_G(v). \tag{2}$$

Thus a column of A may be written as the linear combination of columns of A_G in different ways. So the index function η_A is not uniquely determined by the matrix A itself, instead it is determined by how its columns are expressed as linear combinations of columns of A_G. For simplicity, we use the notation η_A. However, whenever the index function η_A is used, we refer to an explicit expression of its columns as linear combination of columns of A_G which is clear from the context.

The following lemma is well-known (cf. [?]) and follows easily from Equation (1).

Lemma 6.3.3 *Assume that $\sigma : V(G) \cup E(G) \to \{0, 1, \dots\}$ is a mapping which assigns to each vertex and each edge of G a non-negative integer. Then G has a non-singular index function η with $\eta(z) \leq \sigma(z)$ for every $z \in V(G) \cup E(G)$ if and only if there is a square matrix A whose columns are expressed as linear combinations of columns of A_G such that for each $z \in V(G) \cup E(G)$, $\eta_A(z) \leq \sigma(z)$.*

Now we are ready to prove Theorem 6.3.2. Indeed, we shall prove a slightly stronger result.

Theorem 6.3.4 *Assume that G is a connected graph and that F is a spanning tree of G. Then there is a matrix A whose columns are linear combinations of columns of A_G such that* $\text{per}(A) \neq 0$ *and* $\eta_A(v) \leq 1$ *for* $v \in V$, $\eta_A(e) = 0$ *for* $e \in E(F)$ *and* $\eta_A(e) \leq 2$ *for* $e \in E(G-F)$.

Proof. The proof is by contradiction. Assume that the result is not true, and G is a minimum counterexample. It is obvious that $|V| \geq 3$.

Let u be a vertex of G, which is a leaf of F. Assume $N_G(u) = \{u_1, u_2, \ldots, u_k\}$ and orient the edges between u and u_i as $e_i = u_i u \in A(D)$ for $i = 1, 2, \ldots, k$. Assume the edge $u_k u \in E(F)$. Let $G' = G - u$. By the minimality of G, and by Lemma 6.3.3, G' has a valid index function η' such that $\text{per}(A_{G'}(\eta')) \neq 0$, and $\eta'(v) \leq 1$ for $v \in V(G')$, $\eta'(e) \leq 2$ for $e \in E(G')$, and moreover, for $e \in E(F-u)$, $\eta'(e) = 0$.

Let $|E(G')| = m'$ where $m' = |E(G)| - k = m - k$. We view η' as an index function of G, with $\eta'(z) = 0$ if $z \in (V(G) \cup E(G)) - (V(G') \cup E(G'))$. Then $A_G(\eta')$ is an $m \times m'$ matrix, consisting of m' columns of A_G. Let $\eta = \eta'$ except that $\eta(u) = k$. Then $A_G(\eta)$ is an $m \times m$ matrix, which is obtained from $A_G(\eta')$ by adding k copies of the column $A_G(u)$. The added k columns has k rows (the rows indexed by edges incident to u) that are all 1's, and all the other entries of these k columns are 0. Therefore $\text{per}(A_G(\eta)) = \text{per}(A_{G'}(\eta'))k!$, and hence $\text{per}(A_G(\eta)) \neq 0$.

Let $M_0 = A_G(\eta)$. For $i = 1, 2, \ldots, k-1$, if $\eta'(u_i) = 0$, then let $M_i = M_{i-1}$. If $\eta'(u_i) = 1$, then let M_i be obtained from M_{i-1} by replacing $A_G(u_i)$ with $A_G(e_i)$.

Claim 6.3.5 *For* $i = 1, 2, \ldots, k-1$, $\text{per}(M_i) = \text{per}(M_{i-1})$.

Proof. If $\eta'(u_i) = 0$, then $M_i = M_{i-1}$, there is nothing to prove. Assume $\eta'(u_i) = 1$ and M_i is obtained from M_{i-1} by replacing $A_G(u_i)$ with $A_G(e_i)$. Let M'_i be obtained from M_{i-1} by replacing $A_G(u_i)$ with $A_G(u)$. In M'_i, the column $A_G(u)$ occurs $k+1$ times. These $k+1$ columns have k rows (the rows indexed by edges incident to u) that are all 1's, and all the other entries of these $k+1$ columns are 0. Therefore $\text{per}(M'_i) = 0$. By Equation (2), $A_G(e_i) = A_G(u_i) + A_G(u)$. By Equation (1), $\text{per}(M_i) = \text{per}(M_{i-1}) + \text{per}(M'_i) = \text{per}(M_{i-1})$. ∎

Observe that $M_{k-1} = A_G(\eta)$ for an index function η of G for which the following hold:

- $\eta(u_i) = 0$ for $i = 1, 2, \ldots, k-1$, $\eta(u) = k$, $\eta(v) \leq 1$ for other vertices v of G.

- $\eta(e_i) \leq 1$ for $i = 1, 2, \ldots, k-1$, $\eta(e) = 0$ for edges in F, and $\eta(e) \leq 2$ for other edges of G.

By Equation (2), $A_G(u) = A_G(e_i) - A_G(u_i)$ for each $i \in \{1, 2, \ldots, k-1\}$. We replace $k-1$ copies of $A_G(u)$ with $A_G(e_i) - A_G(u_i)$ $(i = 1, 2, \ldots, k-1)$. Denote the resulting matrix by A. The matrix A is indeed the same as $A_G(\eta)$. However, in this new format, we have $\eta_A(v) \leq 1$ for all vertices v of G, and $\eta_A(e) \leq 2$ for all edges of G, and $\eta_A(e) = 0$ for $e \in E(F)$. As $\mathrm{per}(A) = \mathrm{per}(A_G(\eta)) \neq 0$, this completes the proof of Theorem 6.3.4. ∎

Corollary 6.3.6 *Every graph is* $(2, 3)$*-choosable.*

A result slightly stronger than Corollary 6.3.6 follows from Theorem 6.3.4: Suppose G is a connected graph and F is a spanning tree of G. Let $\psi : V(G) \cup E(G) \to \{1, 2, 3\}$ be defined as $\psi(v) = 2$ for every vertex v, $\psi(e) = 1$ for $e \in E(F)$ and $\psi(e) = 3$ for $e \in E(G - F)$. Then G is total weight ψ-choosable. The non-list version of this result was obtained by Kalkowski [**?**].

The following two conjectures, which are weaker than the $(1, 3)$-choosability conjecture and the $(2, 2)$-choosability conjecture respectively, remain open.

Conjecture 6.3.7 *There is a constant k such that every graph with no isolated edges is* $(1, k)$*-choosable.*

Conjecture 6.3.8 *There is a constant k such that every graph is* $(k, 2)$*-choosable.*

Chapter 7

Some other applications

This chapter presents some applications of Combiantorial Nullstellensatz to other fields, such as number theory and geometry.

7.1 Kakeya set

A *Kakeya set* in $\mathbb{R}^n$ is a compact subset K of $\mathbb{R}^n$ that contains a unit line segment of every direction.

It was proved by Besicovitch [4] that the area of a Kakeya set K in R^2 can be arbitrarily small, and K can have Lebesgue measure 0. This implies that for $n \geq 2$, there are Kakeya sets in $\mathbb{R}^n$ with zero n-dimensional Lebesgue measure.

For $E \subseteq \mathbb{R}^n$, the Hausdorff dimension of E, denoted by $d_H(E)$, is defined as follows:

For $\delta > 0$, a δ-cover of E is a family $\{U_i\}$ of finite or countably many subsets $U_i \subseteq \mathbb{R}^n$ of diameter at most δ that covers E, where the *diameter* of a subset U of $\mathbb{R}^n$ is

$$\operatorname{diam}(U) = \sup\{|x - y| : x, y \in U\}.$$

For a positive real number s,

$$\mathcal{H}^s_\delta = \inf\{\sum(\operatorname{diam}(U_i))^s : \{U_i\} \text{ is a } \delta\text{-cover of } E\}.$$

Let

$$\mathcal{H}^s = \lim_{\delta \to 0} \mathcal{H}^s_\delta.$$

Note that

$$\lim_{s\to\infty} \mathcal{H}^s = 0 \text{ and } \lim_{s\to 0} \mathcal{H}^s = \infty.$$

The Hausdorff dimension of E is defined as

$$d_H(E) = \inf\{s : \mathcal{H}^s = 0\} = \sup\{s : \mathcal{H}^s = \infty\}.$$

For any bounded subset E in $\mathbb{R}^n$, it is easy to see that $d_H(E) \le n$, and if E has positive volume, then $d_H(E) = n$. However, for $n \ge 2$, even if E has zero n-dimensional Lebesgue measure, it is still possible that $d_H(E) = n$.

Conjecture 7.1.1 (Kakeya Conjecture) *Any Kakeya set K in $\mathbb{R}^n$ has Hausdorff dimension n.*

This conjecture for $n = 1$ is trivial, for $n = 2$ was proved by Davies [10], for $n = 3$ was proved by Hong Wang and Joshua Zahl [33], and remains open for $n \ge 4$.

Now we consider the version of Kakeya sets in $\mathbb{F}_q^n$.

Definition 7.1.2 *A subset K of $\mathbb{F}_q^n$ ($q = p^m$ for some prime p and positive integer m) is called a Kakeya set if it contains a line in every direction.* [1]

Wolff [34] asked whether there is constant c_n (depends on n only) such that every Kakeya set K in $\mathbb{F}_q^n$ has size at least $c_n q^n$? Dvir proved that $|K| \ge c_n q^{n-1}$. By slightly changing the arguments of Dvir, Alon and Tao proved that $|K| \ge c_n q^n$ [11, 31]. (The bound $|K| \ge c_n q^n$ is proved in [11], which is stronger than Dvir originally proved and is attributed to Alon and Tao).

Theorem 7.1.3 *Every Kakeya set K in $\mathbb{F}_q^n$ has size at least $\binom{n+q-1}{n} \ge \frac{1}{n!} q^n$.*

Proof. Assume to the contrary that there is a Kakeya set K with $|K| < \binom{n+q-1}{n}$.

For any non-negative integer d, denote by $\mathrm{Poly}_{\le d}(\mathbb{F}_q^n)$ the vector space of polynomials $P \in \mathbb{F}_q[x_1, \ldots, x_n]$ of degree at most d. As each $P \in \mathrm{Poly}_{\le d}(\mathbb{F}_q^n)$ is a linear combination of monomials $\prod_{i=1}^n x_i^{t_i}$ with $t_i \ge 0$ and $\sum_{i=1}^n t_i \le d$, and these monomials are linearly independent, we know that $\dim(\mathrm{Poly}_{\le d}(\mathbb{F}_q^n)) = \binom{n+d}{n}$ ($\binom{n+d}{n}$ is the number

[1] I am not sure if this is "the" ananlog of Kakeya set in $\mathbb{F}_q^n$. In $\mathbb{R}^n$, we do not require a Kakeya set to contain a whole line, just a unit length line segment in every direction. Is there something in $\mathbb{F}_p^n$ corresponding to "unit line segment"?

of non-negative integer sequences $(t_1, \ldots, t_n)$ that satisfies $t_1 + \ldots + t_n \leq d$, equivalently, the number of non-negative integer sequences $(t_1, \ldots, t_n, t_{n+1})$ that satisfies $t_1 + \ldots + t_n + t_{n+1} = d$).

Let $\pi : \mathrm{Poly}_{\leq q-1}(\mathbb{F}_q^n) \to \mathbb{F}^{|K|}$ be defined as $\pi(P) = (P(x))_{x \in K}$. Then π is a linear map from the vector space $\mathrm{Poly}_{\leq q-1}(\mathbb{F}_q^n)$ to the vector space $\mathbb{F}^{|K|}$. As

$$\dim(\mathrm{Poly}_{\leq q-1}(\mathbb{F}_q^n)) = \binom{n+q-1}{n} > |K| = \dim(\mathbb{F}^{|K|}),$$

the kernel of π is non-trivial. So there is a non-zero polynomial $P \in \mathrm{Poly}_{\leq q-1}(\mathbb{F}_q^n)$ such that $P(a) = 0$ for all $a \in K$.

Assume P has degree $d \leq q-1$, and let P_d be the d-degree polynomial such that $P - P_d$ has degree at most $d-1$ (i.e., P_d is homogenous polynomial consisting of the highest degree monomials of P). Observe that P cannot be a constant, so $d \geq 1$.

For each $a = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_q^n - \{0\}$, choose $b = (b_1, b_2, \ldots, b_n) \in \mathbb{F}_q^n$ such that $\{at + b : t \in \mathbb{F}_q\} \subseteq K$ (such a point b exists because K contains a line in the direction of a).

Let $Q_a(t) = P(at+b) = P(a_1t + b_1, a_2t + b_2, \ldots, a_nt + b_n)$. As $at + b \in K$ for all $t \in \mathbb{F}_q$, and P vanishes on K, $Q_a(t) = 0$ for all $t \in \mathbb{F}_q$, $Q_a(t) = 0$. The degree of Q_a is the same as the degree of P. So Q_a has degree $d \leq q-1$, this implies that Q_a is the zero polynomial. In particular, the coefficient of t^d is 0. On the other hand, this coefficient is equal to $P_d(a_1, a_2, \ldots, a_n) = P_d(a)$. So $P_d(a) = 0$ for all $a \in \mathbb{F}_q^n - \{0\}$. As $d \geq 1$, we also have $P_d(0) = 0$. Hence $P_d(a) = 0$ for all $a \in \mathbb{F}_q^n$. This is in contradiction to CNS. This completes the proof of Theorem 7.1.3. ∎

The coefficient $\frac{1}{n!}$ in Theorem 7.1.3 is not optimal. It is proved [28] that every Kakeya set K in $\mathbb{F}_q^n$ has size at least $\frac{1}{2}(\frac{q}{2.6})^n$. On the other hand, there are Kakeya sets K in $\mathbb{F}_q^n$ of size $2^{-(n-1)}q^n + O(q^{n-1})$.

7.2 The cap set problem

SET is a pattern-recognition card game invented in 1974 by geneticist Marsha Falco. It is a simple visual puzzle but has a surprisingly deep mathematical structure.

The deck has 81 unique cards. Each card has 4 attributes, each with 3 possible values. The attributes and values are

- Number: 1, 2, or 3.

- Color: red, green, or purple.
- Shape: oval, diamond, or squiggle.
- Shading: solid, striped, or outlined.

A SET is any 3 cards such that, for each of the 4 attributes separately, the values are either all the same, or all different.

For example, the three cards "1 red solid oval, 1 green solid oval, 1 purple solid oval" form a SET: number, shape, shading the same, and color all different.

The three cards "2 red striped diamonds, 2 red striped ovals, 2 red striped squiggles" form a SET: number, color, shading the same, and shape all different.

To play this game, deal 12 cards, face-up in a grid. All players scan simultaneously. First to spot a SET calls "SET!", points to the 3 cards. If correct: take the 3 cards and win 1 point, deal 3 replacements from the deck to keep 12 on the table.

If wrong: then lose 1 point.

If everyone agrees no SET exists in the 12 cards in the grid, deal 3 more cards (now 15, then 18...). When a SET is finally taken, do not replace those extra cards until the table drops back to 12.

Game ends when the deck is empty and no SET remains on the table. Player with the most points wins the game.

We may assume the cards are elements of $\mathbb{F}_3^4$. What referred as a SET is an arithmetic progression of length 3 in the field. This leads to the following problem:

What is the maximum size of a subset of $\mathbb{F}_3^4$ that contains no arithmetic progression of length 3?

More generally, what is the maximum size of a subset A of $\mathbb{F}_3^n$ that contains no arithmetic progression of length 3? Observe that a set A contains no arithmetic progression of length 3 is equivalent to the following statement:

The equation

$$a_1 + a_2 + a_3 = 0$$

has no solution in A^3, except that $a_1 = a_2 = a_3$.

To study this problem, we consider a more general question. Let $\mathbb{F}_q$ be a finite field, and α, β, γ be elements of $\mathbb{F}_q$ with $\alpha + \beta + \gamma = 0$ and $\gamma \neq 0$. Assume A is a subset of $\mathbb{F}_q^n$ such that the equation $\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ has no solution in A^3, except that $a_1 = a_2 = a_3$. What is the maximum size of A?

We denote by M_n the set of monomials in $x_1, \ldots, x_n$ whose degree in each variable is at most $q-1$, and let S_n be the $\mathbb{F}_q$-vector space they span.

Observe that the evaluation map $e : S_n \to \mathbb{F}_q^{\mathbb{F}_q^n}$ given by $e(p) = (p(a))_{a \in \mathbb{F}_q^n}$ is a linear isomorphism. Indeed, both spaces have dimension q^n, and the map e is surjective since for every $a \in \mathbb{F}_q^n$, the polynomial $\prod_{i=1}^n (1 - (x_i - a_i)^{q-1})$ is mapped to the indicator function of point a.

For a real number $d \in [0, (q-1)n]$, let M_n^d be the set of monomials in M_n of degree at most d and S_n^d the subspace of S_n they span. Write m_d for the dimension of S_n^d.

Lemma 7.2.1 *Let $\mathbb{F}_q$ be a finite field, and $A \subseteq \mathbb{F}_q^n$, $\alpha, \beta, \gamma \in \mathbb{F}_q$, $\alpha + \beta + \gamma = 0$. Suppose $P \in S_n^d$, and $P(\alpha a + \beta b) = 0$ for every distinct elements a, b of A. Then $|\{a \in A : P(\gamma a) \neq 0\}| \leq 2m_{d/2}$.*

Proof. Consider the $|A| \times |A|$-matrix B with entries $B_{a,b} = f(\alpha a + \beta b)$ for $a, b \in A$. By our assumption, $B_{a,b} = 0$ if $a \neq b$, i.e., B is a diagonal matrix.

Each entry $f(\alpha x + \beta y) = \sum_{m,m' \in S_n^d, \deg(mm') \leq d} c_{m,m'} m(x) m'(y)$. As the degree of mm' is at most d, at least one of m, m' has degree $\leq d/2$. Thus

$$f(\alpha x + \beta y) = \sum_{m \in S_n^{d/2}} m(x) F_m(y) + \sum_{m' \in S_n^{d/2}} m'(y) G_m(x).$$

For each $m \in S_n^{d/2}$, the $|A| \times |A|$ matrix $C(m)$ with entries $C(m)_{a,b} = m(a) F_m(b)$ is a rank 1 matrix. Similarly, for each $m' \in S_n^{d/2}$, the $|A| \times |A|$ matrix $D(m')$ with entries $D(m')_{a,b} = G_m(a) m'(b)$ is a rank 1 matrix. As

$$B = \sum_{m \in S_n^{d/2}} C(m) + \sum_{m' \in S_n^{d/2}} D(m'),$$

we conclude that the rank of B is at most $2m_{d/2}$. As B is a diagonal matrix, B has at most $2m_{d/2}$ non-zero diagonal entries, i.e., there are at most $2m_{d/2}$ elements $a \in \mathbb{F}_q^n$ such that $f(\alpha a + \beta a) \neq 0$. As $\alpha + \beta = -\gamma$, the lemma is proved. ∎

Theorem 7.2.2 *Let α, β, γ be elements of $\mathbb{F}_q$ with $\alpha + \beta + \gamma = 0$ and $\gamma \neq 0$. Let $A \subseteq \mathbb{F}_q^n$ and the equation $\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ has no solution in A^3, except that $a_1 = a_2 = a_3$. Then $|A| \leq 3m_{(q-1)n/3}$.*

Proof. Let $d \in [0, (q-1)n]$, and let V be the space of polynomials in S_n^d vanishing on the complement of $-\gamma A$. Then V has dimension at least $m_d - (q^n - |-\gamma A|) = m_d - q^n + |A|$.

View the elements of V as functions on $\mathbb{F}_q^n$. Let $P \in V$ be a polynomial with maximal support. Let $\Sigma = \{a \in \mathbb{F}_q^n : P(a) \neq 0\}$ be the support of P. Then $|\Sigma| \geq \dim(V)$, for otherwise, there is a non-zero polynomial $Q \in V$ that vanishes on Σ. Then $P+Q \in V$ has support strictly larger than Σ, in contradiction to the choice of P.

Let $\mathcal{S}(A) = \{\alpha a_1 + \beta a_2, a_1, a_2 \in A, a_1 \neq a_2\}$. By our assumption, $\mathcal{S}(A)$ is disjoint from $-\gamma A$. Hence P vanishes on $\mathcal{S}(A)$. By Lemma 7.2.1, $|\Sigma| \leq 2m_{d/2}$. Therefore

$$m_d - q^n + |A| \leq \dim(V) \leq |\Sigma| \leq 2m_{d/2}.$$

Hence

$$|A| \leq 2m_{d/2} + (q^n - m_d).$$

Note that $q^n - m_d$ is the number of q-power-free monomials whose degree is greater than d. There is a bijection between this polynomials with those monomials of degree less than $(q-1)n-d$: $\prod_{i=1}^n x_i^{t_i} \to \prod_{i=1}^n x_i^{n-1-t_i}$. Hence

$$|A| \leq 2m_{d/2} + m_{(q-1)n-d}.$$

Let $d = 2(q-1)n/3$, we have $|A| \leq 3m_{(q-1)n/3}$. ∎

Let $q = 3$. We estimate $m_{2n/3}$, which is the number of monomials in n variables, each variable has an exponent $0, 1$ or 2, and with total degree at most $2n/3$. Given a triple (a_0, a_1, a_2) such that $a_1 + a_1 + a_3 = n$ and $a_1 + 2a_2 \leq 2n/3$, the number of monomials with a_i variables of degree i for $i = 0, 1, 2$ is $\frac{n!}{a_0!a_1!a_2!}$. Hence

$$m_{2n/3} = \sum_{a_0+a+1a_2=n, a_1+2a_2 \leq 2n/3} \frac{n!}{a_0!a_1!a_2!}.$$

The number of triples (a_0, a_1, a_2) of non-negative integers with $a_0 + a_1 + a_2 = n$ is $\binom{n+3}{3} \approx n^3$. Thus

$$m_{2n/3} \leq n^3 \max\{\frac{n!}{a_0!a_1!a_2!} : a_0 + a + 1a_2 = n, a_1 + 2a_2 \leq 2n/3\}.$$

By Stirling formula, $n! = (1+o(1))\sqrt{2\pi n}n^n e^{-n}$. So $n! \approx n^n e^{-n}$ up to a polynomial factor. Let $a_i = \alpha_i n$, $h(\alpha_0, \alpha_1, \alpha_2) = \sum_{i=0}^2 -\alpha_i \log \alpha_i$, and let

$$\alpha = \max\{h(\alpha_0, \alpha_1, \alpha_2) : \alpha_0 + \alpha_1 + \alpha_2 = 1, \alpha_1 + 2\alpha_2 \leq 2/3\}.$$

Then

$$m_{2n/3} \leq n^3 e^{\alpha n}.$$

By applying the Lagrange multiplier method, we conclude that the maximum of h is attained at

$$\begin{aligned} a_0 &= \frac{32}{3(15+\sqrt{33})}, \\ a_1 &= \frac{4(\sqrt{33}-1)}{3(15+\sqrt{33})}, \\ a_2 &= \frac{(\sqrt{33}-1)^2}{6(15+\sqrt{33})}. \end{aligned}$$

Hence $\alpha \approx 1.0135$. It follows that

$$|A| = O(2.756^n).$$

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